Master 2

Parcours ATIAM

Rapport de stage de fin d'étude

Intégrateurs multisymplectiques par action d'un groupe de Lie: test de méthodes numériques HPC sur des systèmes complètement intégrables.

Étudiant :
Pierre Carré
carre@ircam.fr

Encadrant :
Joël Bensoam
joel.bensoam@ircam.fr

Février – Août 2017

Abstract

We present in this report structure preserving integrators based on the discretization of the Lagrangian field theory. The particular case where the configuration space of the system is a Lie group is then studied to introduce Lie group variational integrators. This type of integrators is applied on the Reissner beam model, and applications for sound synthesis are presented.

Keywords Lie group variational integrators, multisymplectic geometry, structure preserving integrators, Reissner beam.

Résumé

Dans ce rapport sont présentés des intégrateurs numeriques, basés sur la discrétisation de la théorie lagrangienne des champs et préservant la structure des sytèmes. Le cas particulier où l'espace de configuration est un groupe de Lie est ensuite étudié afin d'introduire les intégrateurs variationnels par action d'un groupe de Lie. Ce type d'intégrateur est appliqué au modèle de la poutre de Reissner, et des applications pour la synthèse sonore sont présentées.

Mots-clés Intégrateurs variationels par action d'un groupe de Lie, géométrie multisymplectique, intégrateurs préservant les structures, poutre de Reissner.

Acknowledgement We thank Joël Bensoam for his help and the opportunity he gave us to expose the content of this report before other researchers in the context of the GEOdESiC project, and Thomas Hélie, René Caussé and Robert Piéchaud for their useful feedback.

Contents

Part I
Introduction

1 Context

In many scientific domains such as computational chemistry, weather forecasting or sound synthesis, physical models use time and space scales much smaller than those of the desired output; for example, sound synthesis is ideally performed at sample rates higher than 40kHz over periods of time that exceed several seconds or minutes. Complex physical systems are modelled by Partial Differential Equations (PDEs) whose analytical solutions cannot be determined in general, and are thus approached by numerical methods. The High Performance Computing (HPC) could offer new possibilities for those simulations that demand more and more machine power.

However, the HPC cannot compensate the drawbacks of some computing methods which are useless because of the very nature of some of the physical problems : the modelled physical systems evolve on curved configuration spaces, namely manifolds, making them intrinsically non linear. The non linear effects have a huge impact on the results of the numerical methods, making them irrelevant. Without taking into account their intrinsically non linear structure, the small scale errors percolate to large scale errors and cannot be avoided by simply increasing digital precision.

Algorithms built to preserve the geometrical structure of the physical systems do not present this problem. The so-called geometrical integrators are numerical methods with built-in geometrical properties conservation, which makes them more robust than general purpose methods. Numerous examples have already been deployed in fields from mechanics to financial prediction, that show new possible applications to the HPC domain.

2 Purpose

The main purpose of the internship is to show the interest of geometric integrators for High Performance Computing by building a Lie group multisymplectic integrator, and compare its application on the Reissner beam problem to the associated integrable solutions in order to show its structure preserving features.

The internship is done at the Institut de Recherche et Création Acoustique/Musique (IRCAM) in the S3AM team under te guidance of Joël Bensoam. This production contributes to the elaboration of a project draft, which presents the interest of several mathematical theories for HPC – among which the evoked methods, the Port Hamiltonian Systems (PHS) and the stochastic integrators – in order to obtain a funding from the European Union for the creation of an international consortium of research institutes, lead by IRCAM in the person of Joël Bensoam and Thomas Hélie, and focusing on building frameworks for massively parallel exascale computing.

3 Summary

This report presents the general formalism of Lie group multisymplectic integrators, followed by an application on the Reissner beam problem, that can be used in sound synthesis to model a string.

The formalism of multisymplectic Lagrangian theory is presented in part III. It is the generalisation of the symplectic case presented as an introduction (see part II) to the multisymplectic setting. The discrete counterpart of the discrete multisymplectic Lagrangian formalism is introduced in part IV. The case where the configuration space is a Lie group yields new results for the continuous and discrete formalism, both presented in parts V and VI. Finally, those results are applied in the case of the Reissner beam problem, for which the continuous and discrete models are introduced, a hint on possible integrable solution is given, and applications to sound synthesis are presented.

The continuous formalism of parts III and V are mainly inherited from Bensoam [1] and thus do not constitute a personal work; however, the example on the pendulum in section 8 is by us. Their discrete counterparts are the generalisation of particular applications found in several sources (see Marsden [11] and Vankershaver [15] for example). The Reissner beam discrete model takes elements from Demoures [4].

This work has been presented along with Joël Bensoam to several mathematics and HPC community partners of the GEOdESiC project during a working trip to Edinburgh (Scotland), Groningen (Netherlands) and Trondheim (Norway); this was the occasion to confront this work to other areas, such as PHS and HPC.

We shall point out that the focus is not on the HPC aspects of numerical methods in this report. The subject of applying the presented methods on HPC is a non trivial problem in itself, and we did not have the scientific maturity to tackle it within the six month duration of the internship; this is one of the purposes of the GEOdESiC project.

Part II
Introduction to Lagrangian mechanics and its discretization

We introduce here the approach involved in continuous Lagrangian mechanics, and how to discretize it in order to conserve the structure preserving features.

Section 4 gives the general ideas and philosophy of Lagrangian mechanics in the case where the state of physical systems only varies along time. The goal of this quick introduction is to grasp the main concepts before introducing the more complex case of the Lagrangian field theory which involves the same principles. After introducing the Lagrangian and the action map, we obtain the Euler-Lagrange equations through the Hamilton principle. The two main results of quantity preservation are then proved, namely the symplecticity of the flow and the preservation of momentum maps on solutions.

Section 5 presents the ideas behind the discretization of the Lagrangian mechanics and gives an overview of the current state of the art. The same concepts will be use to build the discretizations in parts IV and VI in the multisymplectic case.

4 Lagrangian mechanics

This exposition is mainly inspired by the beginning of Marsden [12].

A physical system is represented at each time by its position $(q, \dot{q})$ in a tangent bundle TQ with Q an N-dimensional configuration manifold, and let $\pounds : TQ \to \mathbb{R}$ be a Lagrangian associated to the problem. The action map is defined as the time integral of the Lagrangian along a curve :

$$\mathcal{A}(q) := \int_0^T \pounds\,(q, \dot{q})\, dt$$

The Hamilton principle stands that solutions q of the problem are critical points of the action map fixed at the endpoints, that is, for any variation δq of the curve q,

$$\mathrm{d}\mathcal{A}(q) \cdot \delta q = \int_0^T \left(\frac{\partial \pounds}{\partial q^A} - \frac{d}{dt}\frac{\partial \pounds}{\partial \dot{q}^A} \right) \delta q^A dt + \left[\Theta_{\mathcal{L}}(q) \cdot \delta q\right]_0^T = 0 \tag{4.0.1}$$

where $\Theta_{\mathcal{L}} = \frac{\partial \pounds}{\partial \dot{q}^A}$. When the endpoints of the curve q are fixed, the integral on the boundary of the time interval vanishes in (4.0.1), and as the variations δq are arbitrary, we obtain the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial \pounds}{\partial \dot{q}^A} - \frac{\partial \pounds}{\partial q^A} = 0 \tag{4.0.2}$$

which are the equations of motion.

Symplecticity of the flow Let the flow $F_t : TQ \to TQ$ be defined as $F_t(v_q) = (q(t), \dot{q}(t))$ where q is a solution of the Euler-Lagrange equations (4.0.2) such that $(q(0), \dot{q}(0)) = v_q \in TQ$, and let $\mathcal{A}_t : TQ \to \mathbb{R}$ be defined such that

$$\mathcal{A}_t(v_q) := \int_0^t \pounds(F_s(v_q))ds$$

then for any variation δv_q of initial conditions, we obtain

$$\mathrm{d}\mathcal{A}_t(v_q) \cdot \delta v_q = \Theta_\pounds\left(F_s(v_q)\right) \cdot \delta\left(F_s(v_q)\right) - \Theta_\pounds(v_q) \cdot \delta v_q = \left(F_t^* \Theta_\pounds - \Theta_\pounds\right)(v_q) \cdot \delta v_q$$

since the integral on the interior of $[0, t]$ in (4.0.1) vanishes along solutions of Euler-Lagrange equations. Taking the differential of $\mathrm{d}\mathcal{A}_t$ yields

$$0 = \mathrm{d}\mathrm{d}\mathcal{A}_t = F_t^* \mathrm{d}\Theta_{\pounds} - \Theta_{\pounds}$$

or equivalently

$$F_t^* \Omega_{\pounds} = \Omega_{\pounds} \tag{4.0.3}$$

where $\Omega_{\pounds}$ is called the *symplectic form*. This property is called the conservation of the symplectic form by the Lagrangian flow, or *simplecticity of the flow* for short.

Noether theorem Let G be a Lie group leaving the Lagrangian $\pounds$ invariant and acting trivially on time, then the action $\mathcal{A}$ is invariant under the action of the group. This implies the existence of an infinitesimal generator $\xi \in \mathfrak{g}$ of G such that $\xi \lrcorner \mathrm{d}\mathcal{A}_t = 0$ on solutions, hence

$$0 = \xi \lrcorner \left(F_t^* \Theta_{\pounds} - \Theta_{\pounds}\right) = F_t^* \left(\xi \lrcorner \Theta_{\pounds}\right) - \xi \lrcorner \Theta_{\pounds}$$

or equivalently

$$F_t^* \left(\xi \lrcorner \Theta_{\pounds}\right) = \xi \lrcorner \Theta_{\pounds} \tag{4.0.4}$$

The *momentum map* $J_\xi : TQ \to \mathbb{R}$ defined by $J_\xi := \xi \lrcorner \Theta_{\pounds}$ is therefore conserved by the flow, that is, is invariant on solutions of the problem. This is the result of the first Noether theorem.

5 Discretizing the Euler-Lagrange equations

We introduce here the approach that will be used later to obtain the discrete counterpart of Lagrangian mechanics.

One way to obtain a numerical method is to discretize the Euler-Lagrange Ordinary Differential Equations (ODEs) in time using an approximation of time derivatives with finite variations – for example, the first derivative can be approximated by the slope of the line between two consecutive points. This method is straightforward, resolving in an exploitable numerical method. General purpose methods, such as those based on Runge-Kutta, have reached a certain maturity after decades of improvement. However, those give no guarantee on the conservation of invariant quantities, such as momenta or total energy for a closed system.

Another approach is to take a step back and discretize the Lagrangian instead of the Euler-Lagrange equations. In the same way the Euler-Lagrange equation has been obtained from the application of the Hamilton principle to the Lagrangian, its discrete Euler-Lagrange equation counterpart is obtained from the discrete Lagrangian. The knowledge of the Lagrangian allows, both in continuous and discrete settings, to look for geometrical symmetries. The Noether theorem can then be applied, giving some geometrical invariants of the system, yielding useful guarantees on its behaviour. Moreover, the symplectic form is conserved by both continuous and discrete Lagrangian flows. The deduced numerical methods are called structure preserving integrators, and have been actively developed in the 1990's, and are now very well documented (see Marsden [11] for an exposition of the continuous and discrete Lagrangian field theory, and Hairer [7] for a general overview of the properties of structure preserving methods for ODEs).

The two approaches described above can be summed up by the following non-commutating diagram

$$\begin{array}{ccc}
\begin{array}{c}\text{Lagrangian}\\ \pounds : TQ \to \mathbb{R}\end{array} & \longrightarrow & \begin{array}{c}\text{Discrete Lagrangian}\\ \mathcal{L}_d : Q \times Q \to \mathbb{R}\end{array} \\
\downarrow & & \downarrow \\
\text{Euler-Lagrange ODEs} & \longrightarrow & \text{Discrete Euler-Lagrange equations}
\end{array}$$

For a little more than a decade, the extension of this principle in the multisymplectic setting has been studied, that is, when the base space is not restrained to time alone, but also takes into account several spatial dimensions; this should be seen as the generalisation of the previous setting. Similarly, the idea is to obtain the discrete Euler-Lagrange equations by discretizing the Lagrangian and applying the Hamilton principle on it, instead of directly discretizing the Euler-Lagrange PDE. Again, the application of the Noether theorem leads to geometrical invariants in both continuous and discrete settings, and an analogous to the symplectic flow property, the multisymplectic form formula, arises (see Echeverría-Enríquez [13] and Marsden [12] for respectively continuous and discrete multisymplectic settings).

Finally, the particular case where the configuration space is a Lie group allows for a reduction of the Lagrangian and yields the conservation of momenta through Noether theorem in both continuous and discrete settings. The symplectic Lagrangian formalism with Lie groups is well documented (see Holm [8] for an introduction on rigid body dynamics), but the use of Lie groups in Lagrangian field theory and in its discrete counterpart is a much more recent development (see Vankershaver [15] for a general overview of both continuous and discrete multisymplectic settings with Lie groups, and Demoures [4] for a study of the properties of multisymplectic Lie groups variational integrators). This is the approach we will be presenting in parts V and VI, and that will be used to tackle the problem of the Reissner beam in part VII.

Part III
Multisymplectic Lagrangian theory and Lie group symmetries

This part introduces the multisymplectic Lagrangian theory, which is the generalization of the Lagrangian mechanics when the physical system does not only evolve along time, but also along spatial dimensions. The physical invariants of the system, namely the multisymplectic form and the momenta, are characterized. Finally, an example is given in section 8 with the pendulum; it is not multisymplectic, but its simplicity helps to grasp the concepts that will be used in the multisymplectic case on the Reissner beam problem in part VII.

6 Lagrangian field theory

The interest of the Lagrangian field theory is to formulate the partial differential equations describing the system in a geometrical manner, that is, to represent the solution of the equation by a sub-manifold such that the constraints expressed by the equations are translated by the fact that some differential forms cancel on the sub-manifold. This approach yields an intrinsic formulation that may simplify the computation, and also allows to look for equivalences between PDEs by looking for invariants.

6.1 First-order jet bundles

Let M be an orientable smooth manifold, E a smooth manifold and (E, π, M) a principal smooth fibre bundle over M of typical fibre Q. The base M of the fibre bundle represents space-time, and its dimension is $\dim(M) = n + 1$ with n the number of spatial dimensions. In the symplectic case, $n = 0$ and only time is taken into account; multisymplectic case takes place when $n > 0$. Q is the parameter space of the considered physical system, and is of dimension $\dim(Q) = N$. For example, in the case of a rigid body motion, $N = 6$ (3 translations and 3 rotations).

Let $\Gamma(\pi)$ be the set of global sections of the fibre bundle (E, π, M) (also denoted by π). If $\mathcal{U}$ is an open subset of M, we write $\Gamma_{\mathcal{U}}(\pi)$ the set of local sections defined on $\mathcal{U}$. In particular, the section defining the trajectory of a physical system in its configuration space along time belongs to $\Gamma_{\mathcal{U}}(\pi)$ for a given $\mathcal{U}$. From now on, the sections are supposed to be smooth, that is belong to $\mathscr{C}^\infty(E, M)$.

Let (J^1E, π^1, E) be the bundle of 1-jets of sections of π. Roughly speaking, the manifold J^1E will be defined as the space of the derivatives of the parameter space with respect to the base space, which in the simple case of E being position and M time, would be velocity space. For $\mathrm{P} \in E$, $J^1_{\mathrm{P}}E$ is defined by $J^1_{\mathrm{P}}E = \mathrm{preim}_{\pi^1}(\{\mathrm{P}\})$; it is the fibre of J^1E corresponding to P, and its elements are denoted by $\bar{\mathrm{P}}$. Let $\jmath\pi : J^1E \to M$ be defined as $\jmath\pi = \pi \circ \pi^1$; π and π^1 being fibre bundles, this defines a new fibre bundle $(J^1E, \jmath\pi, M)$.

A local coordinate system is defined such that elements of M are noted by a local system $x = (x^\mu)$, $\mu = 1, ..., n + 1$, and those of Q by $y = (y^A)$, $A = 1, ..., N$, where x and y are homomorphisms from M to $\mathbb{R}^{n+1}$ and Q to $\mathbb{R}^N$ respectively. Any element of E is denoted by the local coordinate system (x^μ, y^A). Let $\phi \in \Gamma_{\mathcal{U}}(\pi)$ be a local section of π defined on $\mathcal{U} \subset M$, ϕ is entirely determined by the functions $\phi^A : \mathcal{U} \to F$ such that $\forall x \in \mathcal{U}$, $\phi(x) = (x^\mu, \phi^A(x))$.

In the same manner, a local coordinate system on J^1E can be constructed from (x^μ, y^A) such that any element $\bar{\mathrm{P}}$ of J^1E is denoted by the local coordinates (x^μ, y^A, v^A_μ) where

$$v^A_\mu = \left(\frac{\partial \phi^A}{\partial x^\mu}\right)\bigg|_x$$

with $\pi^1(\bar{\text{P}}) = \text{P}$, $\pi(\text{P}) =$, $y^A = \phi^A(x)$ and $\phi : M \supset \mathcal{U} \to E$ is a representative of $\bar{\text{P}}$. For any given section $\phi : M \supset \mathcal{U} \to E$, the section $j^1\phi : M \supset \mathcal{U} \to J^1E$ is defined so that $\forall x \in \mathcal{U},\ j^1\phi(x) = (x^\mu, y^A, \frac{\partial\phi}{\partial x}\big|_x)$.

Let $TM = \bigcup_{x\in M} T_xM$ be the tangent bundle to M, let $(\vec{\partial}_\mu) \in T_xM$ be a basis of fibre T_xM associated with the dual base $(\mathrm{d}x_\mu) \in T_x^*M$ so that $\mathrm{d}x_\mu(\vec{\partial}_\nu) = \delta^\mu_\nu$. In the same manner, let TQ be the tangent bundle to Q with basis $(\vec{\partial}_A)$, TE the tangent bundle to E with basis $(\vec{\partial}_\mu, \vec{\partial}_A)$ and TJ^1E the tangent vector space to J^1E with basis $(\vec{\partial}_\mu, \vec{\partial}_A, \vec{\partial}^A_\mu)$.

Given any $\bar{\text{P}} = (x^\mu, y^A, v^A_\mu)$, there always exist a section ϕ representative of $\bar{\text{P}}$ such that

$$v^A_\mu = \mathrm{d}y^A \left(\frac{\partial\phi}{\partial x_\mu}\bigg|_x \right).$$

Sections $\psi : M \supset \mathcal{U} \to j^1E$ that have an antecedent ϕ by $j^1\cdot$ so that $\psi = j^1\phi$ are called holonomic sections. The proper definition of that term is the object of section 6.2.

6.2 Contact form

Let $\psi : M \supset \mathcal{U} \to J^1E$ be a holonomic section of $\jmath\pi$, let $\bar{\text{P}} = (x^\mu, y^A, v^A_\mu)$ be a point of $\psi(\mathcal{U})$ and $\vec{u} = \alpha^\mu\vec{\partial}_\mu + \beta^A\vec{\partial}_A + \gamma^A_\mu\vec{\partial}^A_\mu \in T_{\bar{\text{P}}}J^1E$ be a tangent vector to ψ at point $\bar{\text{P}}$, then $\vec{u}$ checks

$$\mathrm{d}y^A(\vec{u}) - v^A_\mu \mathrm{d}x_\mu(\vec{u}) = \beta^A - \alpha^\mu v^A_\mu = 0 \tag{6.2.1}$$

This is the intuitive reason to introduce the contact form, used to characterize the holonomic sections of J^1E.

6.2.1 Contact form

First we define the vertical differential; let $V(\pi) = \mathrm{Ker}\, T\pi$ be the vertical bundle associated with π and $V(\pi^1) = \mathrm{Ker}\, T\pi^1$ the vertical bundle associated with π^1.

Definition 1 (Vertical differential)**.** Let $\phi : \mathcal{U} \to E$ be a section of π, the *vertical differential* of the section ϕ at point $\text{P} \in \phi(\mathcal{U})$ is the map

$$\begin{aligned} \mathrm{d}^V_{\text{P}}\phi : T_{\text{P}}E &\to V_{\text{P}}(\pi) \\ u &\mapsto u - T_{\text{P}}(\phi \circ \pi)u \end{aligned}$$

For any section $\phi = (x^\mu, \phi^A(x^\mu))$ of π and $\forall \text{P} \in \phi(\mathcal{U})$, we have the following results :

$$\mathrm{d}^V_{\text{P}}\phi(\vec{\partial}_\mu) = -\frac{\partial\phi^A}{\partial x^\mu}\bigg|_{\pi(\text{P})} \vec{\partial}_A \tag{6.2.2a}$$

$$\mathrm{d}^V_{\text{P}}\phi(\vec{\partial}_A) = \vec{\partial}_A \tag{6.2.2b}$$

For a given section ϕ, $\mathrm{d}^V_{\text{P}}\phi$ only depends on $j^1\phi(\pi(\text{P})) = \frac{\partial\phi}{\partial x}\big|_{\pi(\text{P})}$; the contact form can now be defined using $\mathrm{d}^V_{\text{P}}\phi$.

Definition 2 (Contact form)**.** Consider $\bar{\text{P}} \in J^1E$, the *contact form* of J^1E is the vectorial 1-form in J^1E defined at point $\bar{\text{P}}$ by

$$\begin{aligned} \theta_{\bar{\text{P}}} : T_{\bar{\text{P}}}J^1E &\to V(\pi) \\ \bar{u} &\mapsto \mathrm{d}^V_{\text{P}}\phi(T_{\bar{\text{P}}}\pi^1(\bar{u})) \end{aligned}$$

with ϕ any representative section of $\bar{\text{P}}$.

The definition of $\theta_{\bar{\text{P}}}$ only depends on $\bar{\text{P}}$ and not on a particular representative section. For any given point $\bar{\text{P}}$, and any section ϕ representative of $\bar{\text{P}}$, the contact form may be evaluated on the base vectors $(\vec{\partial}_\mu, \vec{\partial}_A, \vec{\partial}_\mu^A)$, and, using the equalities (6.2.2a) and (6.2.2b), gives the result

$$\begin{aligned}\theta_{\bar{\text{P}}}(\vec{\partial}_\mu) =& \mathrm{d}^V_{\text{P}}\phi(\vec{\partial}_\mu) = -\left.\frac{\partial \phi^A}{\partial x^\mu}\right|_{\pi(\text{P})} \vec{\partial}_A = -v^A_\mu \vec{\partial}_A \\ \theta_{\bar{\text{P}}}(\vec{\partial}_A) =& \mathrm{d}^V_{\text{P}}\phi(\vec{\partial}_A) = \vec{\partial}_A \\ \theta_{\bar{\text{P}}}(\vec{\partial}^A_\mu) =& \mathrm{d}^V_{\text{P}}\phi(\vec{0}) = \vec{0}.\end{aligned}$$

Hence, let $\vec{u} = \alpha^\mu \vec{\partial}_\mu + \beta^A \vec{\partial}_A + \gamma^A_\mu \vec{\partial}^A_\mu$ be a vector of $T_{\bar{\text{P}}} J^1 E$,

$$\theta_{\bar{\text{P}}}(\vec{u}) = (\beta^A - \alpha^\mu v^A_\mu)\vec{\partial}_A$$

Therefore, the contact form can be written in a local coordinate system

$$\begin{aligned}\theta_{\bar{\text{P}}} &= (\mathrm{d}y^A - v^A_\mu \mathrm{d}x^\mu) \otimes \vec{\partial}_A \\ &=: \theta^A \otimes \vec{\partial}_A\end{aligned}$$

where θ^A is a 1-form equal to $\mathrm{d}y^A - v^A_\mu \mathrm{d}x^\mu$. As we saw previously in (6.2.1), this equation is equal to zero in the case $\vec{u}$ is tangent at point $\bar{\text{P}}$ to an holonomic section ψ representative of $\bar{\text{P}}$; this is used to define holonomic sections.

6.2.2 Holonomic section

Proposition 1. *Let $\psi : M \supset \mathcal{U} \to J^1E$ be a section of $\jmath\pi$, then ψ is holonomic if and only if $\psi^*\theta = 0$.*

Proof. $(\Rightarrow)$: The implication has already been proven in (6.2.1).

$(\Leftarrow)$: Let us assume that $\psi^*\theta = 0$ where $\psi : \mathcal{U} \to J^1E$ is a section of $\jmath\pi$. Consider $x \in \mathcal{U}$, let (x^μ, y^A, v^A_μ) be a system of local coordinates on $\mathcal{U}$, then $\psi(x) = (x^\mu, \phi^A(x^\mu), f^A_\nu(x^\mu))$ and

$$\psi^*\theta = (\mathrm{d}\phi^A - f^A_\mu \mathrm{d}x^\mu)\vec{\partial}_A = \left(\frac{\partial \phi^A}{\partial x^\mu}\mathrm{d}x^\mu - f^A_\mu \mathrm{d}x^\mu\right)\vec{\partial}_A = 0$$

From that can be deduced $f = \frac{\partial \phi}{\partial x}$, which means that $\forall \bar{\text{P}} \in \psi(\mathcal{U})$, $\bar{\text{P}} = (x^\mu, \phi^A(x), \frac{\partial \phi^A}{\partial x^\mu}(x))$, proving that ψ is holonomic. □

For any holonomic section ψ and $\bar{\text{P}} \in \psi(\mathcal{U})$, ψ is tangent to the vector space $\mathrm{Ker}(\theta_{\bar{\text{P}}})$ at point $\bar{\text{P}}$, which has $(\vec{\partial}_\mu + v^A_\mu \vec{\partial}_A, \vec{\partial}^A_\mu)$ as a basis. In other words, $\forall \vec{u} \in T_{\bar{\text{P}}}\psi$ tangent vector to ψ at point $\bar{\text{P}}$ belongs to the kernel of $\theta_{\bar{\text{P}}}$, so that $\theta_{\bar{\text{P}}}(\vec{u}) = \vec{0}$.

6.2.3 Lift of a vector field

In this section, we define the lift of an arbitrary vector field in $\chi(E)$ into the corresponding vector field in $\chi(J^1E)$, so that it transforms holonomic sections into other holonomic sections. To do so, one must determine the properties of a lifted vector field from $\chi(E)$ to $\chi(J^1E)$, that is, given a vector field $Z \in \chi(E)$, determine $j^1 Z \in \chi(J^1E)$.

Proposition 2. *Let $Z = \alpha^\mu \vec{\partial}_\mu + \beta^A \vec{\partial}_A$ be a vector field of $\chi(E)$ with $(\vec{\partial}_\mu, \vec{\partial}_A, \vec{\partial}^A_\mu)$ the natural basis of TJ^1E, the 1-jet prolongation of Z on TJ^1E is the vector field defined at point $\bar{\text{P}} = (x^\mu, y^A, v^A_\mu)$ by*

$$j^1 Z(\bar{\text{P}}) = \alpha^\mu \vec{\partial}_\mu + \beta^A \vec{\partial}_A + \left(\frac{\partial \zeta^A}{\partial x^\mu} + v^B_\mu \frac{\partial \zeta^A}{\partial y^B}\right)\vec{\partial}^A_\mu$$

where $\zeta^A = j^1 Z \lrcorner \theta^A = \beta^A - v^A_\mu \alpha^\mu$.

Proof. The proof is given in appendix B.1. □

6.3 Hamilton principle

The Hamilton principle will be used to determine the trajectories of the studied physical system in J^1E, implying the introduction of the variational formulation. This section is focused on the Hamilton principle and the way it is expressed in the 1-jet bundle formalism.

6.3.1 Lagrangian system

Let $\omega = dx^1 \wedge ... \wedge dx^{n+1} \in \Lambda^{n+1}(M)$ be a fixed volume $(n+1)$-form on the manifold M, the Lagrangian density is defined as a smooth real-valued function $\pounds \in \mathscr{C}^\infty(J^1E)$.

Definition 3 (Lagrangian form)**.** For a local system of coordinates (x^μ, y^A, v^A_μ), the *Lagrangian form* is defined as the $\jmath\pi$-semibasic $(n+1)$-form in J^1E

$$\mathcal{L} = \pounds(x^\mu, y^A, v^A_\mu)\omega$$

Given a bundle and a Lagrangian form, a Lagrangian system can be defined as follows.

Definition 4 (Lagrangian system)**.** A *Lagrangian system* is a pair $((E, \pi, M), \mathcal{L})$ where M is an orientable manifold, (E, π, M) a differentiable bundle and $\mathcal{L}$ a Lagrangian form on E.

Based on the Lagrangian form, the Hamilton principle (also known as principle of stationary action) can be defined, stating that the path of a physical system, represented by a section in J^1E, will minimise the action map.

6.3.2 Hamilton principle

Definition 5 (Hamilton principle)**.** Let $((E, \pi, M), \mathcal{L})$ be a Lagrangian system, $\Gamma_c(\pi)$ the set of compactly supported sections of π, and the action map

$$\begin{aligned} \mathcal{A} : \Gamma_c(\pi) &\to \mathbb{R} \\ \phi &\mapsto \textstyle\int_{\mathcal{U}} (j^1\phi)^* \mathcal{L} \end{aligned} \tag{6.3.1}$$

where $\mathcal{U}$ is the domain of ϕ, the variational problem posed by the Lagrangian form is the problem of searching for the stationary sections of the action map.

In order to apply the Hamilton principle, one need to express the variation of action along an arbitrary vector field $Z \in \chi(E)$; it is given by

$$\delta\mathcal{A} = \int_{\partial\mathcal{U}} (j^1\phi)^* \left(j^1 Z \lrcorner \left(\frac{\partial \pounds}{\partial v^A_\mu} \theta^A \wedge \mathrm{d}^n x_\mu + \mathcal{L} \right) \right) - \int_{\mathcal{U}} (j^1\phi)^* \left(\zeta^A \left(\mathrm{d} \left(\frac{\partial \pounds}{\partial v^A_\mu} \right) - \frac{1}{n+1} \frac{\partial \pounds}{\partial y^A} \mathrm{d}x^\mu \right) \wedge \mathrm{d}^n x_\mu \right) \tag{6.3.2}$$

where $\zeta^A = j^1 Z \lrcorner \theta^A$. The detail of this computation is given in appendix B.2.

This leads to the introduction of the Poincaré-Cartan form.

6.3.3 Euler-Lagrange equations

Definition 6 (Poincaré-Cartan form)**.** Let $\pounds$ be a Lagrangian density, and $\mathcal{L} = \pounds\omega$ a Lagrangian form, the *Poincaré-Cartan form* $\Theta_\mathcal{L}$ is a $(n+1)$-form in J^1E defined by

$$\Theta_\mathcal{L} = \frac{\partial \pounds}{\partial v^A_\mu} \theta^A \wedge \mathrm{d}^n x_\mu + \mathcal{L}$$

We also introduce the Euler-Lagrange 1-form $\mathcal{T}_\mu^A$ defined by

$$\mathcal{T}_\mu^A = \mathrm{d}\left(\frac{\partial \pounds}{\partial v_\mu^A}\right) - \frac{1}{n+1}\frac{\partial \pounds}{\partial y^A}\mathrm{d}x^\mu.$$

Using this notation, the variation of action $\delta\mathcal{A}$ is written

$$\delta\mathcal{A} = \int_{\partial\mathcal{U}} (j^1\phi)^* \left(j^1 Z \lrcorner \Theta_\mathcal{L}\right) - \int_\mathcal{U} (j^1\phi)^* \left(\zeta^A \mathcal{T}_\mu^A \wedge \mathrm{d}^n x_\mu\right) \tag{6.3.3}$$

Since Z is an arbitrary vector field, we can choose one that vanishes on the boundary of the domain, that is $Z(\partial\mathcal{U})$ is identically null. In that case, the first integral of the variation of action is null, and the second integral is equal to zero for any vector field Z, that is for any ζ^A, leading to the Euler-Lagrange equations

$$\forall A \in \{1, ..., N\},\ \left(\frac{\partial}{\partial x^\mu}\frac{\partial \pounds}{\partial v_\mu^A} - \frac{\partial \pounds}{\partial y^A}\right)(j^1\phi) = 0 \tag{6.3.4}$$

To sum up, the Hamilton principle has led to the fact that any solution of the problem verifies the Euler-Lagrange equation (6.3.4).

7 System invariants

7.1 Variation theorem

7.1.1 Lagrangian multisymplectic form

Given the Poincaré-Cartan form $\Theta_\mathcal{L}$, the Lagrangian multisymplectic $(n+2)$-form can be defined as $\Omega_\mathcal{L} = -\mathrm{d}\Theta_\mathcal{L}$; let us compute this expression.

$$\begin{aligned}
\mathrm{d}\Theta_\mathcal{L} &= \mathrm{d}\left(\frac{\partial \pounds}{\partial v_\mu^A}\right) \wedge \theta^A \wedge \mathrm{d}^n x_\mu + \frac{\partial \pounds}{\partial v_\mu^A}\mathrm{d}\theta^A \wedge \mathrm{d}^n x_\mu + \mathrm{d}\mathcal{L} \\
&= \mathcal{T}_\mu^A \wedge \theta^A \wedge \mathrm{d}^n x_\mu + \frac{1}{n+1}\frac{\partial \pounds}{\partial y^A}\mathrm{d}x^\mu \wedge \theta^A \wedge \mathrm{d}^n x_\mu + \frac{\partial \pounds}{\partial y^A}\mathrm{d}y^A \wedge \omega \\
&\quad + \frac{\partial \pounds}{\partial v_\mu^A}\mathrm{d}v_\mu^A \wedge \omega - \frac{\partial \pounds}{\partial v_\mu^A}\mathrm{d}v_\nu^A \wedge \mathrm{d}x^\nu \wedge \mathrm{d}^n x_\mu \\
&= \mathcal{T}_\mu^A \wedge \theta^A \wedge \mathrm{d}^n x_\mu - \frac{\partial \pounds}{\partial y^A}\theta^A \wedge \omega + \frac{\partial \pounds}{\partial y^A}\theta^A \wedge \omega + \frac{\partial \pounds}{\partial v_\mu^A}\mathrm{d}v_\mu^A \wedge \omega - \frac{\partial \pounds}{\partial v_\mu^A}\mathrm{d}v_\mu^A \wedge \omega \\
&= \mathcal{T}_\mu^A \wedge \theta^A \wedge \mathrm{d}^n x_\mu
\end{aligned}$$

Given that $\mathcal{T}_\mu^A$ and θ^A are 1-forms, $-\mathcal{T}_\mu^A \wedge \theta^A = \theta^A \wedge \mathcal{T}_\mu^A$, we get

$$\Omega_\mathcal{L} = \theta^A \wedge \mathcal{T}_\mu^A \wedge \mathrm{d}^n x_\mu \tag{7.1.1}$$

or in coordinates

$$\Omega_\mathcal{L} = \mathrm{d}y^A \wedge \mathrm{d}\left(\frac{\partial \pounds}{\partial v_\mu^A}\right) \wedge \mathrm{d}^n x_\mu + \left(v_\mu^A \mathrm{d}\left(\frac{\partial \pounds}{\partial v_\mu^A}\right) - \frac{\partial \pounds}{\partial y^A}\mathrm{d}y^A\right) \wedge \omega$$

Notice that, since $\zeta^A = j^1 Z \lrcorner \theta^A$ in expression (6.3.3), $\zeta^A \mathcal{T}_\mu^A \wedge \mathrm{d}^n x_\mu = j^1 Z \lrcorner \Omega_\mathcal{L}$. This allows is to express the variation of action as

$$\delta\mathcal{A} = \int_{\partial\mathcal{U}} (j^1\phi)^* \left(j^1 Z \lrcorner \Theta_\mathcal{L}\right) - \int_\mathcal{U} (j^1\phi)^* \left(j^1 Z \lrcorner \Omega_\mathcal{L}\right) \tag{7.1.2}$$

We now have the elements to introduce the variation theorem. This theorem gives a powerful way to investigate for solutions of the Hamiltonian principle by giving equivalent propositions.

7.1.2 Variation theorem

Theorem 1 (Variation theorem)**.** *Let ϕ be a section of π, the following propositions are equivalent*

(1) *ϕ is a stationary point of the action $\mathcal{A}$*

(2) *ϕ is a solution of the Euler-Lagrange equation*

(3) $\forall W \in \chi(J^1E)$, $(j^1\phi)^*(W \lrcorner \Omega_{\mathcal{L}}) = 0$

Proof. Only the implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are discussed here.

(1) $\Rightarrow$ (2) This is a direct result from the computing of the variation of action in section 6.3 that led us to the Euler-Lagrange equation (6.3.4).

(2) $\Rightarrow$ (3) Let ϕ be a section of π and $W \in \chi(J^1E)$, where ϕ is a solution of the Euler-Lagrange equation.

Let $X_\nu \in \chi(J^1E)$ be the vector field $X_\nu = Tj^1\phi(\vec{\partial}_\nu)$, and let X be the $(n+1)$-vector field $X = (X_1, ..., X_{n+1})$ tangent to $j^1\phi$, then

$$(j^1\phi)^*(W \lrcorner \Omega_{\mathcal{L}})(\vec{\partial}_1, ..., \vec{\partial}_{n+1}) = (W \lrcorner \Omega_{\mathcal{L}})(X)$$

Given the fact that $\theta^A(X_\nu) = 0$ because X_ν is tangent to $j^1\phi$,

$$\begin{aligned}(W \lrcorner \Omega_{\mathcal{L}})(X) &= \Omega_{\mathcal{L}}(W, X) = (\theta^A \wedge \mathcal{T}^A_\mu \wedge \mathrm{d}^n x_\mu)(W, X)\\ &= \theta^A(W)(\mathcal{T}^A_\mu \wedge \mathrm{d}^n x_\mu)(X) + (-1)^\nu \theta^A(X_\nu)(\mathcal{T}^A_\mu \wedge \mathrm{d}^n x_\mu)(W, X_{\sim\nu})\\ &= \theta^A(W)(\mathcal{T}^A_\mu \wedge \mathrm{d}^n x_\mu)(X)\end{aligned}$$

where $X_{\sim\nu} = (X_1, ..., X_{\nu-1}, X_{\nu+1}, ..., X_{n+1})$.

In the particular case where W is tangent to $j^1\phi$, then obviously $\theta^A(W) = 0$ since W verifies the holonomic criteria. If $W \in V(\pi^1)$, then it has no components along $\vec{\partial}_\mu$ nor $\vec{\partial}_A$, hence $\theta^A(W) = 0$. In either case

$$(W \lrcorner \Omega_{\mathcal{L}})(X) = 0$$

Note that ϕ does not need to be a stationary point of $\mathcal{A}$ to prove the point in those cases.

In the general case where $W \in \chi(J^1E)$, ϕ satisfies the Euler-Lagrange equation

$$(j^1\phi)^* \left(\mathcal{T}^A_\mu \wedge \mathrm{d}^n x_\mu\right) = 0$$

This yields

$$(j^1\phi)^*(\mathcal{T}^A_\mu \wedge \mathrm{d}^n x_\mu)(\vec{\partial}_1, ..., \vec{\partial}_{n+1}) = (\mathcal{T}^A_\mu \wedge \mathrm{d}^n x_\mu)(X) = 0$$

and finally

$$\Omega_{\mathcal{L}}(W, X) = \theta^A(W)(\mathcal{T}^A_\mu \wedge \mathrm{d}^n x_\mu)(X) = 0$$

which proves the implication. □

7.2 Multisymplectic geometry

In this section, we discuss the covariant Hamiltonian field theory, which is the counterpart of the Lagrangian field theory, and its relation to the symplectic manifold structure. After a short introduction to Hamiltonian systems, we introduce the properties of the symplectic form relative to the Hamiltonian formalism. We then explore the properties of the multisymplectic form in the Lagrangian field theory.

7.2.1 Covariant Hamiltonian formalism

Given a Lagrangian system $(J^1, \mathcal{L})$, we define its dual Hamiltonian system $(J^1E^*, \mathcal{H})$ thanks to the fibre-preserving map $\mathbb{F}\mathcal{L} : J^1E \to J^1E^*$ called the *Legendre transform*, which to any given element of J^1E denoted in local coordinates by (x^μ, y^A, v^A_μ) associates the element (x^μ, y^A, p^A_μ) where p^A_μ are called the *conjugate momenta* and are expressed by

$$p^A_\mu = \frac{\partial \pounds}{\partial v^A_\mu}.$$

The relation between v^A_μ and p^A_μ is a continuously differentiable bijection. To the Lagrangian density $\pounds$ corresponds the covariant Hamiltonian $\mathbb{H}$ expressed by

$$\mathbb{H} = \frac{\partial \pounds}{\partial v^A_\mu} v^A_\mu - \pounds$$

and the Hamiltonian form $\mathcal{H}$ is given by the product of $\mathbb{H}$ with the volume form ω.

The $(n+1)$-form $\Theta_\mathcal{H}$ is defined on the dual jet bundle J^1E^* so that the Poincaré-Cartan $(n+1)$-form is the pull back of $\Theta_\mathcal{H}$ by the Legendre transform on J^1E^*; $\Omega_\mathcal{H}$ is defined analogously with respect to the Lagrangian pre-multisymplectic $(n+2)$-form $\Omega_\mathcal{L}$, yielding

$$\Theta_\mathcal{L} = \mathbb{F}\mathfrak{L}^*\Theta_\mathcal{H}$$
$$\Omega_\mathcal{L} = \mathbb{F}\mathfrak{L}^*\Omega_\mathcal{H}$$

In coordinate, we have

$$\Theta_\mathcal{H} = p^A_\mu \mathrm{d}y^A \wedge \mathrm{d}^n x_\mu - \mathbb{H}\omega$$
$$\Omega_\mathcal{H} = \mathrm{d}y^A \wedge \mathrm{d}p^A_\mu \wedge \mathrm{d}^n x_\mu + \mathrm{d}\mathbb{H} \wedge \omega$$

We introduce the canonical multisymplectic $(n+2)$-form on J^1E^* $\Omega^\mathcal{H}_{can} := \Omega^\mathcal{H}_\mu \wedge \mathrm{d}^n x_\mu = \mathrm{d}y^A \wedge \mathrm{d}p^A_\mu \wedge \mathrm{d}^n x_\mu$; this relates to $\Omega_\mathcal{H}$ thanks to the relation $\Omega_\mathcal{H} = \Omega^\mathcal{H}_{can} + \mathrm{d}\mathbb{H} \wedge \omega$. The canonical mutlisymplectic $(n+2)$-form on J^1E^* can now be defined by pulling back $\Omega^\mathcal{H}_{can}$ by the Legendre transform, such that $\Omega^\mathcal{L}_{can} := \mathbb{F}\mathfrak{L}^*\Omega_{can}\mathcal{H}$. We obtain the expression

$$\begin{aligned}\Omega^\mathcal{L}_{can} &= \mathrm{d}y^A \wedge \left(\frac{\partial^2 \pounds}{\partial x^\nu \partial v^A_\mu}\mathrm{d}x^\nu + \frac{\partial^2 \pounds}{\partial y^B \partial v^A_\mu}\mathrm{d}y^B + \frac{\partial^2 \pounds}{\partial v^B_\nu \partial v^A_\mu}\mathrm{d}v^B_\nu\right) \wedge \mathrm{d}^n x_\mu \\ &= \frac{\partial^2 \pounds}{\partial x^\mu \partial v^A_\mu}\mathrm{d}y^A \wedge \omega + \frac{\partial^2 \pounds}{\partial y^B \partial v^A_\mu}\mathrm{d}y^A \wedge \mathrm{d}y^B \wedge \mathrm{d}^n x_\mu + \frac{\partial^2 \pounds}{\partial v^B_\nu \partial v^A_\mu}\mathrm{d}y^A \wedge \mathrm{d}v^B_\nu \wedge \mathrm{d}^n x_\mu\end{aligned}$$

Before we move on to the multisymplectic properties of the Lagrangian systems, we shall first discuss the easier case of symplectic manifolds and Hamiltonian systems.

7.2.2 Symplecticity of the flow

In the symplectic setting, we only consider a one dimensional base space M, namely time; a number of important results then arise for Hamiltonian systems that we shall point out. In the next section, the multisymplectic setting will be presented; this reason of the current section is partly to introduce some complex notions in this simplified setting that may be easier to manipulate and that have a more intuitive physical meaning than their multisymplectic counterparts.

Definition 7 (Symplectic manifold)**.** A *symplectic manifold* (X, Ω) is a smooth manifold X equipped with a closed non-degenerate 2-form Ω called the symplectic form.

Let us consider a base space M of dimension 1 that we identify with time, and a configuration space Q of dimension N, then the cotangent bundle TQ^* of the dual jet bundle J^1E^* is the phase space of dimension $2N$ whose elements are denoted (q,p) where $q \in Q$ and $p := \frac{\partial \pounds}{\partial \dot{q}}$ is the conjugate momentum of $\dot{q} := \frac{\partial q}{\partial t}$. The smooth manifold TQ^* equipped with the 2-form Ω_{can} is a symplectic manifold. Indeed, Ω_{can} is a $(n+2)$-form, where $n+1 = 1 = \dim(M)$, and since $\mathrm{d}\Omega_{can} = \mathrm{d}\left(\mathrm{d}q^A \wedge \mathrm{d}p^A\right) = 0$, $\mathrm{d}\Omega_{can}$ is closed.

Definition 8 (Symplectic map)**.** A differentiable map $g : \mathcal{U} \to X$, where $\mathcal{U} \subset X$ is an open set, is *symplectic* with respect to the symplectic manifold (X, Ω) if

$$g^*\Omega = \Omega$$

Let us illustrate this notion on the simple example of linear mappings in the case of the previously defined Hamiltonian symplectic manifold. For any $\xi = (\xi_q, \xi_p), \nu = (\nu_q, \nu_p)$

$$\Omega_{can}(\xi,\nu) = \mathrm{d}q^A \wedge \mathrm{d}p^A(\xi,\nu) = \mathrm{d}q^A(\xi^q)\mathrm{d}p^A(\nu^p) - \mathrm{d}p^A(\xi^p)\mathrm{d}q^A(\xi^q) = \xi_q^A\nu_p^A - \nu_q^A\xi_p^A = \xi^T J\nu$$

where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. In the case where $N = 1$, then $\Omega_{can}(\xi,\nu) = \det \begin{bmatrix} \xi^q & \nu^q \\ \xi^p & \nu^p \end{bmatrix}$ is the oriented area of the parallelogram defined by ξ and ν; in the general case, it is the sum of the oriented areas of the projections onto the planes (q^A, p^A) of the N-dimensional parallelogram. Now, let $A : TQ^* \to TQ^*$ be a linear mapping such that $\forall x \in TQ^*,\ A(x) = Ax$, and let us suppose that this map is symplectic, then $\forall \xi, \nu$,

$$\begin{aligned}(A^*\Omega_{can})(\xi,\nu) &= \Omega_{can}(A\xi, A\nu) = \xi^T A^T JA\nu \\ &= \Omega_{can}(\xi,\nu) = \xi^T J\nu\end{aligned}$$

This condition is true for all ξ and ν if $A^T JA = J$. In the case where $N = 1$, symplecticity of the linear mapping is synonymous with area preservation (see figure 1); in fact, this property is not limited to linear mappings and can be shown for any symplectic mapping (see section 8 for an illustration of this property on the Lagrangian flow of the pendulum, and Hairer [7] for a complete proof).

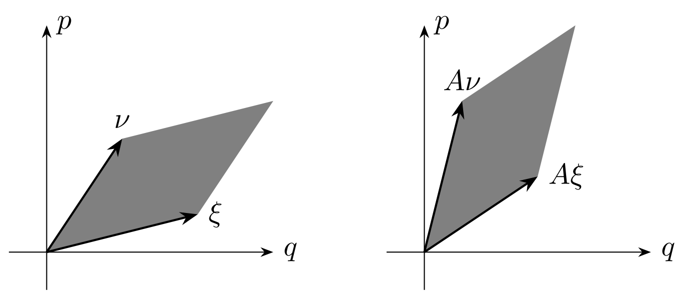


Figure 1 – Simplecticity of a linear mapping in the case $N = 1$.

In this setting, the Lagrangian canonical symplectic form is expressed by

$$\Omega_{can}^{\mathcal{L}} = \frac{\partial^2 \pounds}{\partial t \partial v_\mu^A}\mathrm{d}y^A \wedge \mathrm{d}t + \frac{\partial^2 \pounds}{\partial y^B \partial v^A}\mathrm{d}y^A \wedge \mathrm{d}y^B + \frac{\partial^2 \pounds}{\partial v^B \partial v^A}\mathrm{d}y^A \wedge \mathrm{d}v^B \qquad (7.2.1)$$

Notice that $\Omega_{can}^{\mathcal{L}}$ is not completely defined on TQ, on the contrary to $\Omega_{can}^{\mathcal{H}}$ who belongs to the phase space, since a term with $\mathrm{d}t$ appeared. This makes the Lagrangian canonical symplectic form less intuitive to represent, even in the case where $N = 1$.

We shall see here an important property of the Lagrangian flow in the symplectic configuration, and then, thanks to an analogous reasoning, how this property is expressed in the multisymplectic configuration

Definition 9 (Lagrangian flow)**.** The *flow of a Lagrangian system* over time is the map that for any initial conditions $(q_0, \dot{q}_0)$ at time t_0 associates the solution $(q(t), \dot{q}(t))$ of the Hamiltonian system at time t,

$$\begin{aligned} F^t_{\mathcal{L}} : TQ &\to TQ \\ (q_0, \dot{q}_0) &\mapsto (q(t), \dot{q}(t)) \end{aligned}$$

where $(q(t_0), \dot{q}(t_0)) = (q_0, \dot{q}_0)$.

Proposition 3. *The flow of a solution of a Lagrangian system is symplectic relative to the symplectic form $\Omega_{\mathcal{L}}$ defined by* (7.1.1)*.*

Proof. Let us consider the action map (6.3.1), it is written in our setting

$$\mathcal{A}(q) = \int_{t_0}^{t_1} \pounds(q(t), \dot{q}(t))\mathrm{d}t.$$

Taking its variation relative to δq yields in accordance with (6.3.3)

$$\mathrm{d}\mathcal{A}(q) \cdot (\delta q) = \int_{t_0}^{t_1} \left(\frac{\partial \pounds}{\partial q^A} - \frac{\partial}{\partial t}\frac{\partial \pounds}{\partial \dot{q}^A} \right) (q(t), \dot{q}(t))\delta q^A(t)\mathrm{d}t + \left[\Theta_{\mathcal{L}}(q(t), \dot{q}(t)) \cdot \delta q(t)\right]_{t_0}^{t_1}.$$

Let $(q, \dot{q})$ be such that $(q(0), \dot{q}(0)) = (q_0, \dot{q}_0) =: v_q \in TQ$, we define the restricted action map $\mathcal{A}_t : TQ \to \mathbb{R}$ such that

$$\mathcal{A}_t(v_q) := \int_{t_0}^{t_1} \pounds(F^s_{\mathcal{L}}(v_q))\mathrm{d}t$$

where $F^s_{\mathcal{L}} = (q(s), \dot{q}(s))$. Evaluated along a solution of the Lagrangian problem, the variation of action becomes

$$\mathrm{d}\mathcal{A}_t(v_q) \cdot (\delta v_q) = \Theta_{\mathcal{L}}(F^t_{\mathcal{L}}(v_q)) \cdot \delta(F^t_{\mathcal{L}}(v_q)) - \Theta_{\mathcal{L}}(v_q) \cdot \delta v_q = \left(F^{t*}_{\mathcal{L}}\Theta_{\mathcal{L}} - \Theta_{\mathcal{L}}\right)(v_q) \cdot \delta v_q$$

From this we get $\mathrm{d}\mathcal{A}_t = F^{t*}_{\mathcal{L}}\Theta_{\mathcal{L}} - \Theta_{\mathcal{L}}$, and thus $\mathrm{dd}\mathcal{A}_t = F^{t*}_{\mathcal{L}}\Theta_{\mathcal{L}} - \Theta_{\mathcal{L}} = 0$; since $\Omega_{\mathcal{L}} = -\mathrm{d}\Theta_{\mathcal{L}}$, this yields

$$F^{t*}_{\mathcal{L}}\Omega_{\mathcal{L}} = \Omega_{\mathcal{L}}.$$

The Lagrangian flow of a solution of the Lagrangian system is indeed symlectic with respect to the Lagrangian mulitsymplectic 2-form $\Omega_{\mathcal{L}}$ □

7.2.3 Multisymplectic form formula

The multisymplectic form formula is the analogous of the symplecticity of the Lagrangian flow (property 3) in the case of multisymplectic field theory.

Theorem 2 (Multisymplectic form formula)**.** *Let ϕ be a solution of the Lagrangian problem defined on $\mathcal{U}$, then for any $Y, Z \in \chi(E)$,*

$$\int_{\partial\mathcal{U}} (j^1\phi)^*(j^1Y \lrcorner j^1Z \lrcorner \Omega_{\mathcal{L}}) = 0 \tag{7.2.2}$$

Proof. Starting from the action variation expression (7.1.2), we write

$$\delta\mathcal{A} = \int_{\partial\mathcal{U}} (j^1\phi)^* \left(j^1Z \lrcorner \Theta_{\mathcal{L}}\right) - \int_{\mathcal{U}} (j^1\phi)^* \left(j^1Z \lrcorner \Omega_{\mathcal{L}}\right)$$

We recall from the variation theorem that any solution of the Lagrangian system cancels the second integral. A first variation of a solution ϕ of the system is a vertical vector field $Y \in \chi(E)$ whose flow

F_ε^Y is such that $F_\varepsilon^Y \circ \phi$ also is a solution of the Lagrangian system. In other words, using the variation theorem, $\forall Z \in \chi(E)$,

$$j^1(F_\varepsilon^Y \circ \phi)^*(j^1Z \lrcorner \Omega_\mathcal{L}) = 0$$

Taking the derivative with respect to ε yields

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \left(j^1(F_\varepsilon^Y \circ \phi)^*(j^1Z \lrcorner \Omega_\mathcal{L})\right) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left(j^1(\phi^* F_\varepsilon^Y)^*(j^1Z \lrcorner \Omega_\mathcal{L}) - j^1\phi^*(j^1Z \lrcorner \Omega_\mathcal{L})\right) \\ &= (j^1\phi)^* \left(\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left(F_\varepsilon^{Y*}(j^1Z \lrcorner \Omega_\mathcal{L}) - (j^1Z \lrcorner \Omega_\mathcal{L})\right)\right) \\ &= (j^1\phi)^* L_{j^1Y} j^1Z \lrcorner \Omega_\mathcal{L} = 0\end{aligned}$$

We then show that, for any ϕ solution of the Euler-Lagrange equations,

$$\int_{\partial\mathcal{U}} (j^1\phi)^*(j^1Y \lrcorner j^1Z \lrcorner \Omega_\mathcal{L}) = 0$$

□

7.3 Lagrangian symmetries and Noether theorem

A *symmetry of a Lagrangian system* is defined as a diffeomorphism in the phase space of the system that leaves the Lagrangian form invariant. This diffeomorphism can be thought of being generated by a vector field S, leading to the introduction of the notion of infinitesimal natural symmetry.

Definition 10 (Infinitesimal natural symmetry)**.** Let $((E,\pi,M),\mathcal{L})$ be a Lagrangian system, an *infinitesimal natural symmetry* of the Lagrangian system is a vector field $S \in \chi(E)$ such that its prolongation j^1S leaves $\mathcal{L}$ invariant, that is

$$L_{j^1S}\mathcal{L} = 0$$

In the case where S is an infinitesimal symmetry of the Lagrangian, the Poincaré-Cartan form is also left invariant by S.

Proposition 4. *If $S \in \chi(E)$ is an infinitesimal symmetry of the Lagrangian system $((E,\pi,M),\mathcal{L})$, then*

$$L_{j^1S}\Theta_\mathcal{L} = 0$$

Proof. We refer to Echeverría-Enríquez [13] for the proof. □

The first Noether's theorem states that the existence of symmetries leads to the conservation of quantities called currents.

Theorem 3 (First Noether's theorem)**.** *Let $S \in \chi(E)$ be an infinitesimal natural symmetry of the Lagrangian system $((E,\pi,M),\mathcal{L})$, then the n-form $J(S) = (j^1S) \lrcorner \Theta_\mathcal{L}$ is a constant closed form on the critical sections ϕ of the variational problem, that is the* Noether current $(j^1\phi)^*J(S)$ *is a conserved quantity.*

Proof. Let $\phi \in \Gamma(\mathcal{U})$ be a stationary point of the action, for any $W \in \chi(J^1E)$,

$$(j^1\phi)^*(W \lrcorner \mathrm{d}\Theta_\mathcal{L}) = 0$$

according to theorem 1; it is true in particular for $W = j^1S$. Moreover, by invariance of the Lagrangian with respect to S, we have by proposition (4)

$$L_{j^1S}\Theta_\mathcal{L} = 0$$

This yields

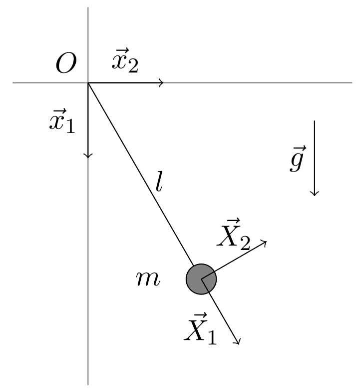
$$\begin{aligned}(j^1\phi)^* L_{j^1S}\Theta_{\mathcal{L}} &= (j^1\phi)^* \left(\mathrm{d}((j^1S)\lrcorner\Theta_{\mathcal{L}})\right) + (j^1\phi)^* \left((j^1S)\lrcorner\mathrm{d}\Theta_{\mathcal{L}}\right) \\ &= (j^1\phi)^* \left(\mathrm{d}((j^1S)\lrcorner\Theta_{\mathcal{L}})\right) = \mathrm{d}\left((j^1\phi)^* \left((j^1S)\lrcorner\Theta_{\mathcal{L}}\right)\right) \\ &= 0\end{aligned}$$

which proves the theorem. □

8 Example : the pendulum

In this section, we shall illustrate the developed formalism in the easiest case, the symplectic setting with a 1 dimensional fibre.

Consider a simple pendulum, consisting of a punctual mass m attached to an inextensible massless rod of length l moving in a fixed plane which origin is at the pivotal point of the rod, and only consider as forces acting on the mass the gravitational force and the tension in the rod. Let $(\vec{X}_1, \vec{X}_2)$ be a direct base of normalised orthogonal vectors invariant in the frame of the pendulum, such that $\vec{X}_1$ is along the rod towards the mass, and also define $(\vec{x}_1, \vec{x}_2)$ as a base of the fixed Galilean referential with the same orientation as $(\vec{X}_1, \vec{X}_2)$ and with $\vec{x}_1$ along $\vec{g}$ the gravitational field; let denote the position of the mass at time t by $X(t)$ and $x(t)$ respectively in the moving frame and in the fixed frame; at the equilibrium



$$x_{eq} = X = \begin{bmatrix} l \\ 0 \end{bmatrix}.$$

Let θ be the angle between the string and the vertical line, oriented by $(\vec{x}_1, \vec{x}_2)$; at any time t, x and X check

$$x(t) = R_{\theta(t)}X$$

with $R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \in SO(2)$ the rotation of angle θ that belong to the special orthogonal group of order 2. Since Q is a commutative Lie group, we identify $(SO(2), \times)$ with the more trivial $(\mathbb{R}, +)$, and use θ instead of R_θ.

We define the base space $M = \mathbb{R}$ as the time axis, and $(E = M \times Q, \pi, M)$ the (trivial) fibre bundle of typical fibre $Q = \mathbb{R}$ the parameter space in which θ evolves. There exists a global coordinate system such that for any $\mathrm{P} \in E$, $\mathrm{P} = (t, \theta)$. From this we get that the 1-jet of ϕ in $J^1E = M \times TQ$ is given in coordinates by $j^1\phi(t) = (t, \theta(t), \dot{\theta}(t))$.

The Lagrangian density is defined as the difference between the kinetic and potential energies

$$\pounds(\theta, \omega) = \frac{1}{2}ml^2\omega^2 + mgl\cos(\theta).$$

The associated Lagrangian system is given by the pair $((E, \pi, M), \mathcal{L})$ where $\mathcal{L}(\theta, \dot{\theta}) = \pounds(\theta, \dot{\theta})\mathrm{d}t$ is the Lagrangian 1-form. The Euler-Lagrange equation is given for any section ϕ of π at point $\mathrm{P} = (t, \theta)$ by

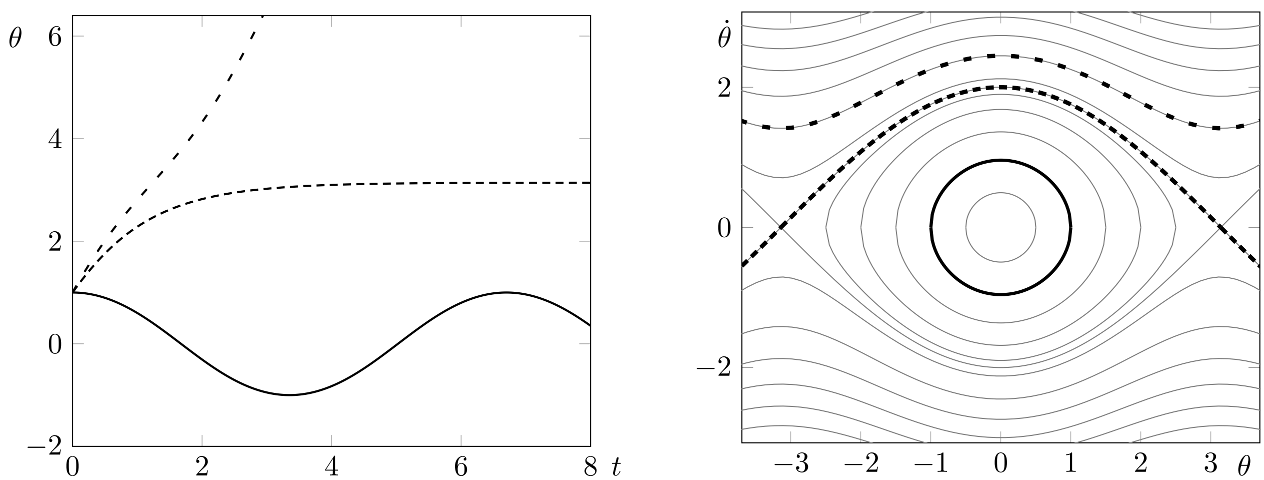
$$\begin{aligned}&\left(\frac{\partial}{\partial t}\frac{\partial \pounds}{\partial \omega} - \frac{\partial \pounds}{\partial \theta}\right)(t, \theta(t), \dot{\theta}(t)) = 0 \\ \Leftrightarrow\quad &\ddot{\theta}(t) + \frac{g}{l}\sin(\theta(t)) = 0. \qquad (8.0.1)\end{aligned}$$

As states the variational theorem, any solution verifying the Hamilton principle, *i.e.* a stationary point of the action, is solution of the Euler-Lagrange equation. Represented in the tangent space, the solutions follow a path corresponding to a constant level energy, represented as gray trajectories evolving in the counter trigonometric direction in figure 2.

The pendulum system is completely integrable, and its solution can be expressed depending on the total energy of the system. Let the mechanical energy $E(\theta, \omega)$ be defined as

$$E(\theta, \omega) = \frac{1}{2} m l^2 \omega^2 + mgl(1 - \cos(\theta))$$

The explicit expressions are given in appendix B.3.



(a) θ represented against time

(b) Solutions on the tangent space TQ

Figure 2 – Three solutions of the problem corresponding to different levels of energy.

We can check the equivalence between the Euler-Lagrange equation (8.0.1) and the last proposition in the variation theorem 1. For this, we compute the Poincaré-Cartan 1-form

$$\Theta_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial \omega}(\mathrm{d}\theta - \omega \mathrm{d}t) + \mathcal{L} = ml^2 \left(\frac{g}{l} \cos(\theta) - \frac{1}{2}\omega^2 \right) \mathrm{d}t + ml^2 \omega \mathrm{d}\theta.$$

We also compute the multisymplectic form

$$\begin{aligned} \Omega_{\mathcal{L}} &= (\mathrm{d}\theta - \omega \mathrm{d}t) \wedge \left(\mathrm{d}\left(\frac{\partial \mathcal{L}}{\partial \omega} \right) - \frac{\partial \mathcal{L}}{\partial \theta} \mathrm{d}t \right) = (\mathrm{d}\theta - \omega \mathrm{d}t) \wedge (ml^2 \mathrm{d}\omega + mgl \sin(\theta) \mathrm{d}t) \\ &= -mgl \sin(\theta) \mathrm{d}t \wedge \mathrm{d}\theta + ml^2 \mathrm{d}\theta \wedge \mathrm{d}\dot{\theta} + \dot{\theta} ml^2 \mathrm{d}\dot{\theta} \wedge \mathrm{d}t \end{aligned} \tag{8.0.2}$$

Let $W = \alpha \vec{\partial}_t + \beta \vec{\partial}_\theta + \gamma \vec{\partial}_\omega \in \chi(J^1 E)$ be a vector field of $J^1 E$, then

$$W \lrcorner \Omega_{\mathcal{L}} = ml^2 \left(-\alpha \left(\frac{g}{l} \sin(\theta) \mathrm{d}\theta + \omega \mathrm{d}\omega \right) + \beta \left(\frac{g}{l} \sin(\theta) \mathrm{d}t + \mathrm{d}\omega \right) + \gamma \left(\omega \mathrm{d}t - \mathrm{d}\theta \right) \right) \tag{8.0.3}$$

Let ϕ be a section of π, then the pullback of (8.0.3) by $j^1\phi$ is equal to

$$(j^1\phi)^*(W \lrcorner \Omega_\ell) = ml^2 \left(-\alpha \left(\frac{g}{l} \sin(\theta)\dot{\theta} + \dot{\theta} \frac{\partial \dot{\theta}}{\partial t} \right) + \beta \left(\frac{g}{l} \sin(\theta) + \frac{\partial \dot{\theta}}{\partial t} \right) \right) \mathrm{d}t \tag{8.0.4}$$

The term in γ is null because $j^1\phi$ is a holonomic section, hence the pullback of the contact form θ by $j^1\phi$ is identically null. By independence of α and β in (8.0.4), the condition $(j^1\phi)^*(W \lrcorner \Omega_{\mathcal{L}}) = 0$ gives us the two equations

$$\dot{\theta}\left(\ddot{\theta} + \frac{g}{l}\sin(\theta)\right) = 0 \tag{8.0.5a}$$

$$\ddot{\theta} + \frac{g}{l}\sin(\theta) = 0 \tag{8.0.5b}$$

The equation (8.0.5a) is implied by (8.0.5b), which is the Euler-Lagrange equation (8.0.1); the equivalence of the propositions of the variational theorem 1 is indeed verified in the case of the pendulum problem.

Since the pendulum evolves in the symplectic setting in a 1-dimensional fibre, the conservation of the Lagrangian flow can be illustrated. We recall the proposition (3) that stated the symplecticity of the Lagrangian flow relative to the symplectic form $\Omega_{\mathcal{L}}$ defined by (7.1.1), which in the case of the pendulum is given by (8.0.2). As can be seen in the expression, the component of $\Omega_{\mathcal{L}}$ dual to TQ is a constant 2-form, and since the Lagrangian flow has no component along time, $\Omega_{\mathcal{L}}$ is proportional to the measure of oriented areas in TQ. For this reason, the result of the symplecticity of the Lagrangian flow is equivalent to the area preservation of elements of TQ by the flow, as shown in 7.2. Figure 3 shows the time evolution of two figures; the area preservation by the Lagrangian flow can clearly be observed. Note that this property is not true in general, it only occurs if the term associated to $\mathrm{d}y \wedge \mathrm{d}v$ in the expression of $\Omega^{\mathcal{L}}_{can}$ in (7.2.1), namely $\frac{\partial^2 \pounds}{\partial^2 v}$, is constant.

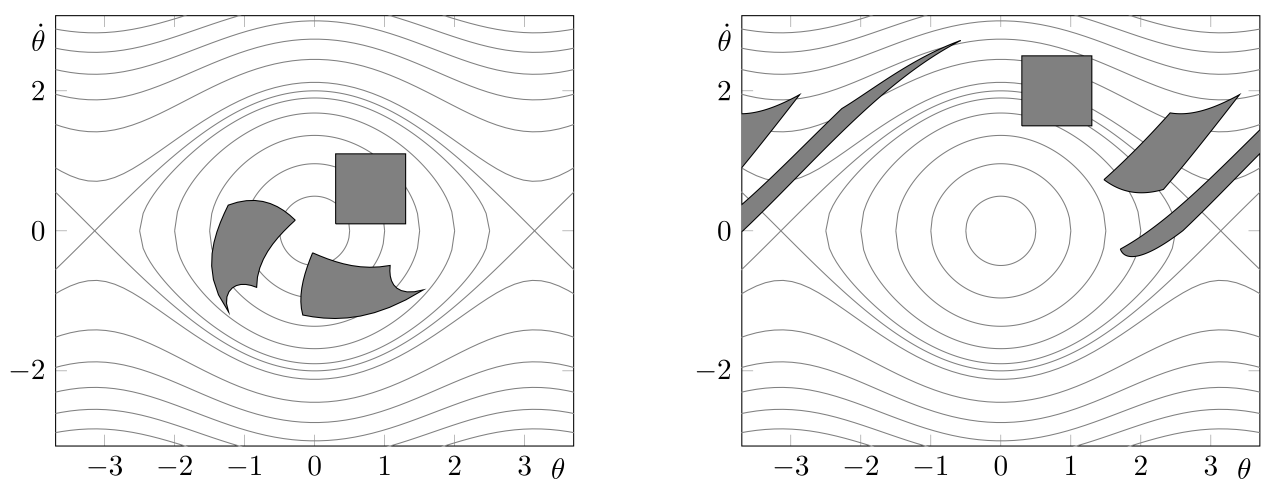


Figure 3 – Area preservation of squares by the Lagrangian flow.

Part IV
Discrete multisymplectic Lagrangian theory and Lie group symmetries

The motivation of this part is to obtain numerical methods that preserve geometric properties of the system. As explained in section 4, general methods based on the direct discretization of the Euler-Lagrange PDEs have no such properties, but reproducing the development of part III on a discrete system equipped with a discrete Lagrangian allows the definition of discrete analogous to the continuous objects, and ensures their conservation by the discrete methods.

9 Discrete Lagrangian field theory

9.1 Base space discretization

In order to build the discrete version of the Lagrangian field theory, we first need to discretize the base space. There are several ways to do this, such as defining a mesh over the base space, for example the regular mesh consisting of rectangles in the case of a two dimensional base space; in a more general approach, we will discretize the base space a set of non degenerates polytopes in M.

We call k-polytope σ a polytope with k corners, and we denote its associated set of vertices by $[v_{i_1}, ..., v_{i_k}]$, where $v_i \in M$ for any i. We define the notation $\sigma_j := v_{i_j}$ where $\sigma = [v_{i_1}, ..., v_{i_j}, ..., v_{i_k}]$. Let us define the set of non degenerate polytopes $\mathcal{P}_M$ on M. From now on we suppose that all the polytopes have the same number of corners k, and we denote the set of polytopes $\mathcal{P}_M^k$. We define the *set of boundary indices* the set $\mathcal{F}(\mathcal{I}) \subset \mathcal{I}$ such that $\forall i \in \mathcal{F}(\mathcal{I})$, $v_i \cap \partial\mathcal{U} = \emptyset$. Finally, let $n : V \to I$ be the global index bijection such that for any vertex it associates a unique index : $\forall p \in \{1, ..., k\}, \forall (i_j)_{1\leq j\leq k} \in \mathcal{I}$, $n([v_{i_1}, ..., v_{i_k}]_p) = i_p$. The use of a global index map n may be overly complicated, but as we shall see in the computation of the variation of action, it is a very efficient way to deal with the sum indexes.

9.2 Hamilton principle

The overall idea of this section is to compute the variation of action on a discrete Lagrangian. Such a Lagrangian is defined as the integral of the continuous Lagrangian on a portion of the base space M, here on a polytope; it is therefore a discrete differential form, which is homogeneous to a "portion" of action in this case.

Theorem 4. *Let $\mathcal{L}$ be a Lagrangian, let σ be a $(n+1)$-polytope of M with k vertices, $k > n+1$, and $(q_1, ..., q_k) \in Q^k$, if $|\sigma|$ and $\max_{1\leq i<j\leq k} \|q_j - q_i\|$ are small enough, there exist an* exact discrete Lagrangian *$\mathcal{L}_E$ defined by*

$$\mathcal{L}_E(\sigma_1, q_1, ..., \sigma_k, q_k) = \int_\sigma (j^1\phi)^*\mathcal{L}$$

where $j^1\phi$ is the unique solution of the Euler-Lagrange equation for $\mathcal{L}$ which satisfies $\forall i \in \{1, ..., k\}$, $\phi(\sigma_i) = q_i$.

Proof. We refer to [11] for the proof. □

Another way to present the discrete Lagrangian is done by Leok [9]

$$\mathcal{L}_E(\sigma_1, q_1, ..., \sigma_k, q_k) = \operatorname{ext}_{\substack{\phi\in\Gamma(\sigma)\\ \phi(\sigma_i)=q_i}} \int_\sigma (j^1\phi)^*\mathcal{L}$$

which is equivalent, but shows more clearly that the discrete Lagrangian is defined as the evaluation of the action on the solution of the Hamilton principle.

The link between this definition of the discrete Lagrangian, whose arguments are a ploytope and a set of points in E, and the continuous Lagrangian, defined on J^1E, may not seem obvious at first sight, but is a common formulation in discrete differential calculus, and allows to easily treat the discrete equivalents of $\Theta_{\mathcal{L}}$ and $\Omega_{\mathcal{L}}$ among others. However, a classical way to discretize the Lagrangian is to define a grid on the base space, and replace the exact derivatives of ϕ by approximations based on finite differences, and relates more intuitively to the continuous formalism. This type of approach is used in the example of the pendulum in section 11.

We now introduce the notion of *discrete path*, which is the analogous of the 1-jet of a section of E defined in section 6. It lacks the notion of derivative since it is not defined with the help of a contact form, but as we just explained, the idea of derivation can be retrieved from the definition of the discrete Lagrangian, within which hides the approximation of the derivatives, and thus the approximate derivatives can be deduced from the particular choice of a discrete Lagrangian.

Definition 11 (Discrete path)**.** A *discrete path* ψ_d in $E = M \times Q$ is defined on a set $\mathcal{I} \in \mathbb{N}$ by

$$\begin{aligned} \psi_d : \mathcal{I} &\to E \\ i &\mapsto (v(i), \phi_d(i)) \end{aligned}$$

where $v : \mathcal{I} \to M$ and $\phi_d : \mathcal{I} \to Q$. The space of discrete path is denoted by $\mathcal{C}_d$.

Given a discrete path ψ_d defined on $\mathcal{I} = \{1, ..., N\}$, we define an *associated discrete curve* $q : \{v(1), ..., v(N)\} \to Q$ such that $\forall i \in \mathcal{I}$, $q(v(i)) = \phi_d(i)$, and an *associated set of vertices* $(v_i)_{i \in \mathcal{I}}$ such that $\forall i \in \mathcal{I}$, $v_i = v(i)$. Let us define the set of non degenerate polytopes $\mathcal{P}_M$ on M associated to ψ_d as the set of polytopes whose corners correspond to the set of vertices ψ_d. For notational purpose, we will write $\psi_d(\sigma) := (\psi_d(n(\sigma_0)), ..., \psi_d(n(\sigma_k)) = (\sigma_0, \phi(\sigma_0), ..., \sigma_k, \phi(\sigma_k))$.

The *exact discrete action* $\mathcal{A}_d^E$ for ψ_d is defined by

$$\mathcal{A}_d^E(\psi_d) = \sum_{\sigma \in \mathcal{P}_M^k} \mathcal{L}_E(\psi_d(\sigma)).$$

Let us consider a section $\phi \in \Gamma(\mathcal{U})$ and $\psi_d \in \mathcal{C}$ a discrete path on $\mathcal{U}$, and denote $\forall i \in \mathcal{I}$, $q_i := q(v_i) = \phi(v_i)$. It can easily be proved that, when verifying the hypothesis of theorem 4, the exact discrete action evaluated on the discrete section ψ_d is equal to the action evaluated on the continuous section ϕ solution to the Euler-Lagrange equations :

$$\begin{aligned} \mathcal{A}_d^E(\psi_d) &= \sum_{\sigma \in \mathcal{P}_M^k} \mathcal{L}_E(\psi_d(\sigma)) = \sum_{\sigma \in \mathcal{P}_M^k} \mathcal{L}_E(\sigma_1, \phi(\sigma_1), ..., \sigma_k, \phi(\sigma_k)) \\ &= \sum_{\sigma \in \mathcal{P}_M^k} \int_\sigma (j^1\phi)^* \mathcal{L} = \int_{\mathcal{U}} (j^1\phi)^* \mathcal{L} = \mathcal{A}(\phi) \end{aligned}$$

As arises in the definition of the discrete exact Lagrangian, we need to know the exact solution $j^1\phi$ of the Euler-Lagrange equation on each $(n+1)$-polytope σ in order to know the discrete exact Lagrangian on those polytopes. Obviously, this is the goal of the problem, and for that reason we have to approach the discrete exact Lagrangian $\mathcal{L}_E$ with an discrete approximated Lagrangian $\mathcal{L}_d$, or simply discrete Lagrangian, whose expression only relies on ψ_d. This is done by replacing the infinite dimensional space of section on each polytope $\Gamma(\sigma)$ by a finite dimensional subspace satisfying the boundary conditions on ϕ.

Definition 12 (Discrete Lagrangian)**.** Let $\mathcal{L}_E$ be a discrete exact Lagrangian, a *discrete Lagrangian* $\mathcal{L}_d : E^k \to \mathbb{R}$ of order r is defined such that there exist an open subset $\mathcal{V} \in E^k$ and constants $C_0^{\mathcal{V}}, ..., C_n^{\mathcal{V}} > 0$ and $h_0^{\mathcal{V}}, ..., h_n^{\mathcal{V}} > 0$, and constant integers $r_0, ..., r_n \geq r$, so that for any $(n+1)$-polytope σ with k vertices such that $\forall i \in \{1, ..., k\}$, $\max_{1 \leq m < n \leq k} \|pr_i(\sigma_n) - pr_i(\sigma_m)\| := h_i \leq h_i^{\mathcal{V}}$,

$$\|\mathcal{L}_d\left(\psi(\sigma)\right) - \mathcal{L}_E\left(\psi(\sigma)\right)\| \leq \sum_{i=0}^{n} C_i^{\mathcal{V}} h_i^{r_i+1}$$

for all solutions $j^1\phi$ of the Euler-Lagrange equations with boundary condition $\phi(\sigma_i) \in \mathcal{V}$.

Along with the discrete Lagrangian $\mathcal{L}_d$ we define the *discrete action* $\mathcal{A}_d$ for ψ_d as

$$\mathcal{A}_d(\psi_d) = \sum_{\sigma \in \mathcal{P}_M^k} \mathcal{L}_d(\psi_d(\sigma)).$$

As for the continuous action, finding the solutions to the discrete problem is equivalent to looking for the stationary solutions of the discrete action map. In order to compute the variation of discrete action, we first recall the notion of ith derivative defined for any $f : X^k \to \mathbb{R}$ by $D_i f(x_1, ..., x_k) = \frac{\partial f}{\partial x_i}(x_1, ..., x_k)$.

Let $\psi_d \in \mathcal{C}_d$ be a discrete path, and $\delta\psi_d \in T_{\psi_d}\mathcal{C}_d$ an arbitrary variation of ψ_d such that $\forall i \in \mathcal{I}$, $(\delta\psi_d)_i = (\delta v_i, \delta q_i)$ for the purpose of notation, then the variation of action at ψ_d in direction $\delta\psi_d$ is expressed by

$$\begin{aligned}
\delta\mathcal{A}_d(\psi_d) := \mathrm{d}\mathcal{A}_d|_{\psi_d}(\delta\psi_d) &= \sum_{\sigma \in \mathcal{P}_M^k} \mathrm{d}\mathcal{L}_d(\psi_d(\sigma))(\delta\psi_d) \\
&= \sum_{\sigma \in \mathcal{P}_M^k} \sum_{i=1}^{k} D_{2i-1}\mathcal{L}_d(\psi_d(\sigma))\mathrm{d}v_{n(\sigma_i)}(\delta\psi_d) + D_{2i}\mathcal{L}_d(\psi_d(\sigma))\mathrm{d}q_{n(\sigma_i)}(\delta\psi_d) \\
&= \sum_{\sigma \in \mathcal{P}_M^k} \sum_{i=1}^{k} D_{2i-1}\mathcal{L}_d(\psi_d(\sigma))\delta v_{n(\sigma_i)} + D_{2i}\mathcal{L}_d(\psi_d(\sigma))\delta q_{n(\sigma_i)} \\
&= \sum_{j \in \mathcal{I}} \sum_{\substack{\sigma \in \mathcal{P}_M^k \\ \sigma_i = v_j}} D_{2i-1}\mathcal{L}_d(\psi_d(\sigma))\delta v_j + D_{2i}\mathcal{L}_d(\psi_d(\sigma))\delta q_j \\
&= \sum_{j \in \mathcal{I} \setminus \mathcal{F}(\mathcal{I})} D_{\mathrm{DEL}}\mathcal{L}_d|_{v_j}\left((\delta\psi_d)_j\right) + \sum_{\substack{\sigma \in \mathcal{P}_M^k \\ \sigma \cap \partial\mathcal{U} \neq \emptyset}} \sum_{\substack{i \in \{1,...,k\} \\ \sigma_i \in \partial\mathcal{U}}} \Theta^i_{\mathcal{L}_d}(\psi_d(\sigma))
\end{aligned}$$

where the *discrete Euler-Lagrange map* $D_{\mathrm{DEL}}\mathcal{L}_d$ is defined for any vertex v in the interior of $\mathcal{P}_M^k$ by

$$D_{\mathrm{DEL}}\mathcal{L}_d|_v = \sum_{\substack{\sigma \in \mathcal{P}_M^k \\ \sigma_i = v}} D_{2i-1}\mathcal{L}_d(\psi_d(\sigma))\mathrm{d}v + D_{2i}\mathcal{L}_d(\psi_d(\sigma))\mathrm{d}q$$

and the *discrete Poincaré-Cartan 1-forms* $\Theta^i_{\mathcal{L}_d}$ are defined by

$$\Theta^i_{\mathcal{L}_d}(\psi_d(\sigma)) = D_{2i-1}\mathcal{L}_d(\psi_d(\sigma))\mathrm{d}v_{n(\sigma_i)} + D_{2i}\mathcal{L}_d(\psi_d(\sigma))\mathrm{d}q_{n(\sigma_i)}$$

In the case of variations $\delta\psi_d$ vanishing at the end points, ψ_d is a solution of the Hamilton problem if and only if the discrete Euler-Lagrange map is null on the interior vertices of $\mathcal{P}_M^k$, which leads to the discrete Euler-Lagrange equations expressed for any $j \in I \backslash \mathcal{F}(I)$ by

$$\sum_{\substack{\sigma \in \mathcal{P}_M^k \\ \sigma_i = v_j}} D_{2i-1}\mathcal{L}_d(\psi_d(\sigma)) = 0 \tag{9.2.1a}$$

$$\sum_{\substack{\sigma \in \mathcal{P}_M^k \\ \sigma_i = v_j}} D_{2i}\mathcal{L}_d(\psi_d(\sigma)) = 0 \tag{9.2.1b}$$

In the continuous case, those two equations are both expressed in the Euler-Lagrange equation, which gives only one equation to verify; but discrete paths solutions of the first equation are not automatically solutions of the second equation, since they are independent in the general case.

The discrete Poincaré-Cartan forms verify

$$\mathrm{d}\mathcal{L}_d = \sum_{i=1}^{k} \Theta^i_{\mathcal{L}_d}.$$

The *discrete Cartan 2-forms* are defined as $\Omega^i_{\mathcal{L}_d} = -\mathrm{d}\Theta^i_{\mathcal{L}_d}$. Given that $\mathrm{d}(\mathrm{d}\mathcal{L}_d) = 0$, the discrete Cartan forms verify

$$\sum_{i=1}^{k} \Omega^i_{\mathcal{L}_d} = 0$$

10 System invariants

10.1 Multisymplectic geometry

10.1.1 Symplecticity of the discrete flow

In the symplectic setting, we define the *discrete Lagrangian flow* as the iteration of the numerical method on a pair $(q_0, q_1) \in Q \times Q$.

Definition 13 (Discrete flow)**.** The *discrete flow* of a discrete Lagrangian system over time is the map that, for any initial conditions (q_0, q_1) at time t_0 associates the solution (q_N, q_{N+1}) of the discrete Lagrangian system at time t_N,

$$\begin{aligned} F^{t_N}_{\mathcal{L}_d} : Q \times Q &\to Q \times Q \\ (q_0, q_1) &\mapsto (q_N, q_{N+1}) \end{aligned}$$

Proposition 5. *The discrete flow of a solution of the discrete Lagrangian problem is symplectic with respect to the discrete symplectic form* $\Omega_{\mathcal{L}_d}$*.*

Proof. We refer to Marsden [11] for the proof. □

In the symplectic case, one can can be convinced of the existence of the flow by noticing that the polytopes are in this case segments, and that therefore the Euler-Lagrange equations involve the vertices on the boundary of two joined segments, that is on three consecutive indexes.

10.1.2 Discrete multisymplectic form formula

Theorem 5 (Discrete multisymplectic form formula)**.** *Let* ψ_d *be a solution of the discrete Euler-Lagrange equations, then for any* $Y, Z \in \chi(E_d)$*,*

$$\sum_{\substack{\sigma \in \mathcal{P}_M^k \\ \sigma \cap \partial\mathcal{U} \neq \emptyset}} \sum_{\substack{i \in \{1,\ldots,k\} \\ \sigma_i \in \partial\mathcal{U}}} \left(\psi_d^*(j^1 Y \lrcorner j^1 Z \lrcorner \Omega^i_{\mathcal{L}_d})\right)(\sigma) = 0 \tag{10.1.1}$$

Proof. When taking the double differential of the discrete action $\mathcal{A}_d$ evaluated at a solution ψ_d along first variations Y and Z, one obtain that

$$\mathrm{dd}\,\mathcal{A}_{d\,\psi_d}(Y,Z) = \mathrm{d}\left(\sum_{\substack{\sigma\in\mathcal{P}_M^k\\ \sigma\cap\partial\mathcal{U}\neq\emptyset}}\sum_{\substack{i\in\{1,\dots,k\}\\ \sigma_i\in\partial\mathcal{U}}}\Theta^i_{\mathcal{L}_d}(\psi_d(\sigma))\right)(Y,Z)$$

and since $\mathrm{dd}\mathcal{A}_d = 0$, the result of theorem (5) is proven. □

10.2 Lagrangian symmetries and Noether's theorem

We here use the notation proposed by Marsden [11] and Demoures [4].

Let G be a Lie group with the group action $\Phi^E : G \times E \to E$ on E that acts trivially on the base space, let us define the lift to the discrete jet space for any $g \in G$ by $\Phi_g^{J^1_dE}(\psi_d(\sigma)) = (\Phi_g^E(\psi_d(\sigma_1)), \dots, \Phi_g^E(\psi_d(\sigma_k)))$ for any σ and ψ_d; the corresponding infinitesimal generator is defined by $\xi^{J^1_dE}(\psi_d(\sigma)) = (\xi^E(\psi_d(\sigma_1)), \dots, \xi^E(\psi_d(\sigma_k)))$. A discrete Lagrangian is said to be invariant under the lifted action if for any $g \in G$, $\mathcal{L}_d \circ \Phi_g^{J^1_dE} = \mathcal{L}_d$. The group G is then a symmetry of $\mathcal{L}_d$, and this implies that $\mathcal{L}_d$ is also infinitesimally invariant.

Definition 14 (Discrete infinitesimal natural symmetry)**.** Let $((E,\pi,M),\mathcal{L}_d)$ be a Lagrangian system, a *discrete infinitesimal natural symmetry* of the Lagrangian system is a vector field $\xi \in \mathfrak{g}$ leaving $\mathcal{L}_d$ invariant, that is

$$\xi \lrcorner \mathrm{d}\mathcal{L}_d = 0$$

The discrete momentum maps $J^i_{\mathcal{L}_d} : J^1_dE :\to \mathfrak{g}^*$, $i \in \{1, \dots, k\}$ are defined for all $\xi^E \in \mathfrak{g}$ by $J^i_{\mathcal{L}_d} \cdot \xi^E := \xi^{J^1_dE} \lrcorner \Theta^i_{\mathcal{L}_d}$. For any path ψ_d solution of the discrete Euler-Lagrange equations and $\sigma \in \mathcal{P}^k_M$,

$$\left(\sum_{i=1}^k J^i_{\mathcal{L}_d}\cdot\xi^E\right)(\psi_d(\sigma)) = \left(\sum_{i=1}^k \xi^{J^1_dE}\lrcorner\Theta^i_{\mathcal{L}_d}\right)(\psi_d(\sigma)) = \left(\xi^{J^1_dE}\lrcorner\mathrm{d}\mathcal{L}_d\right)(\psi_d(\sigma)) = 0$$

This yields

$$\left(\sum_{i=1}^k J^i_{\mathcal{L}_d}\cdot\xi^E\right)(\psi_d(\sigma)) = 0 \tag{10.2.1}$$

This is the statement of the local discrete Noether theorem.

To obtain the global discrete Noether theorem, the discrete momentum maps are summed over an arbitrary subdomain of $\mathcal{P}^k_M$. For this purpose, we define $\mathcal{J}_\mathcal{S}$ where $\mathcal{S} \subset \mathcal{P}^k_M$ such that

$$\mathcal{J}_\mathcal{S}(\psi_d) = \sum_{\sigma\in\mathcal{S}}\left(\sum_{i=1}^k J^i_{\mathcal{L}_d}\cdot\xi^E\right)(\psi_d(\sigma))$$

Theorem 6 (Discrete global Noether first theorem)**.** *Let $\xi^{J^1_dE}$ be an infinitesimal symmetry of the discrete Lagrangian $\mathcal{L}_d$, let ψ_d be a solution of the Lagrangian system defined by $\mathcal{L}_d$, then for any $\mathcal{S} \subset \mathcal{K}$,*

$$\mathcal{J}_\mathcal{S}(\psi_d) = 0 \tag{10.2.2}$$

Proof. The proof is obtained immediately by summing (10.2.1) over $\sigma \in \mathcal{S}$. □

11 Example : the pendulum

A solution of the problem is now a path $\psi_d = (t_i, \theta_i)$, $i \in \{0, ..., N\}$. Let us first consider the case where $t_i = ih$ with $h \in \mathbb{R}_+^*$ is the time step. We choose to define the discrete Lagrangian by

$$\mathcal{L}_d(\theta_0, \theta_1) = h\pounds\left(\theta_0, \frac{\theta_1 - \theta_0}{h}\right) = h\left(\frac{1}{2}\left(\frac{\theta_1 - \theta_0}{h}\right)^2 + \cos(\theta_0)\right)$$

where the derivative $\dot{\theta}(0)$ has been approximated by $\frac{\theta_1 - \theta_0}{h}$. Here we chose to take $m = g = l = 1$ without loss of generality to lighten the equations. This Lagrangian is at least of order 1 in time, since

$$\begin{aligned}\mathcal{L}_E(t_0, \theta_0, t_1, \theta_1) &= \int_{t_0}^{t_1} \pounds\left(\theta, \dot{\theta}\right) dt = \int_{t_0}^{t_1} \pounds\left(\theta_0, \frac{\theta_1 - \theta_0}{h}\right) + \mathcal{O}(h) dt \\ &= h\pounds\left(\theta_0, \frac{\theta_1 - \theta_0}{h}\right) + \mathcal{O}(h^2) = \mathcal{L}_d(\theta_0, \theta_1) + \mathcal{O}(h^2)\end{aligned}$$

Computing one term further would show the term in h^2 is non zero, hence the Lagrangian is of order exactly 1.

By applying (9.2.1b) on the discrete Lagrangian, we obtain the discrete Euler-Lagrange equation $\forall k \in \{1, ..., N-1\}$

$$\frac{\theta_{k+1} - 2\theta_k + \theta_{k-1}}{h^2} + \sin(\theta_k) = 0 \qquad (11.0.3)$$

This numerical method is also known as the symplectic Euler method. We also compute $\Theta^i_{\mathcal{L}_d}$ and $\Omega_{\mathcal{L}_d}$

$$\begin{aligned}\Theta^1_{\mathcal{L}_d}(\theta_0, \theta_1) &= \left(-\frac{\theta_1 - \theta_0}{h} + h\sin(\theta_0)\right) \mathrm{d}\theta_0 \\ \Theta^2_{\mathcal{L}_d}(\theta_0, \theta_1) &= \frac{\theta_1 - \theta_0}{h} \mathrm{d}\theta_1 \\ \Omega_{\mathcal{L}_d}(\theta_0, \theta_1) &= \frac{1}{h} \mathrm{d}\theta_0 \wedge \mathrm{d}\theta_1\end{aligned}$$

The discrete flow is given $\forall k \in \{1, ..., N-1\}$ by

$$F_{\mathcal{L}_d}(\theta_{k-1}, \theta_k) = (\theta_k, 2\theta_k - \theta_{k-1} - h^2\sin(\theta_k))$$

It is indeed symplectic relative to $\Omega_{\mathcal{L}_d}$ since

$$\begin{aligned}\left(F^*_{\mathcal{L}_d}\Omega_{\mathcal{L}_d}\right)(\theta_{k-1}, \theta_k) &= \frac{1}{h}\mathrm{d}\theta_k \wedge \mathrm{d}\left(2\theta_k - \theta_{k-1} - h^2\sin(\theta_k)\right) = \frac{-1}{h}\mathrm{d}\theta_k \wedge \mathrm{d}\theta_{k-1} \\ &= \frac{1}{h}\mathrm{d}\theta_{k-1} \wedge \mathrm{d}\theta_k = \Omega_{\mathcal{L}_d}(\theta_{k-1}, \theta_k)\end{aligned}$$

which we recall is the analogous of the symplectic form formula in the symplectic setting.

Having in mind to compare the properties of symplectic integrators with more general methods, we introduce the explicit Euler method; it is obtained by the direct discretization of the Euler-Lagrange equations for the pendulum. First, we introduce

$$y := \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} \qquad \dot{y} = \begin{bmatrix}\dot{\theta} \\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta} \\ -\sin(\theta)\end{bmatrix}$$

and we discretize as

$$y_{k+1} = \begin{bmatrix}\theta_{k+1} \\ \dot{\theta}_{k+1}\end{bmatrix} = \begin{bmatrix}a(y_k, y_{k+1}) \\ b(y_k, y_{k+1})\end{bmatrix}.$$

In the case where a (or b) depends explicitly on y_{k+1}, the obtained method is said to be *implicit*, otherwise it is *explicit*. If we define a and b such that

$$a(y_k y_{k+1}) = \theta_k + h\dot{\theta}_k \qquad b(y_k, y_{k+1}) = \dot{\theta}_k - h\sin(\theta_{k+1})$$

we find our previously defined symplectic Euler method; we note that the computation of θ_k is explicit, and $\dot{\theta}_k$ implicit. In the case both are explicit, we obtain the explicit Euler method

$$a(y_k y_{k+1}) = \theta_k + h\dot{\theta}_k \qquad b(y_k, y_{k+1}) = \dot{\theta}_k - h\sin(\theta_k)$$

which is also of order 1.

The property of symplecticity of the flow can be observed on the tangent space TQ as the conservation of oriented areas, as explained in section 7.2, and confronted to results for other methods. The figure 4 shows the transportation of a square by the discrete flows of the explicit Euler and symplectic Euler methods. The explicit Euler method is clearly not symplectic, since the area of the square grows bigger, whereas the area of the square transported by the symplectic Euler method seems constant (up to a certain order of approximation of $\dot{\theta}$), as predicted by the theory.

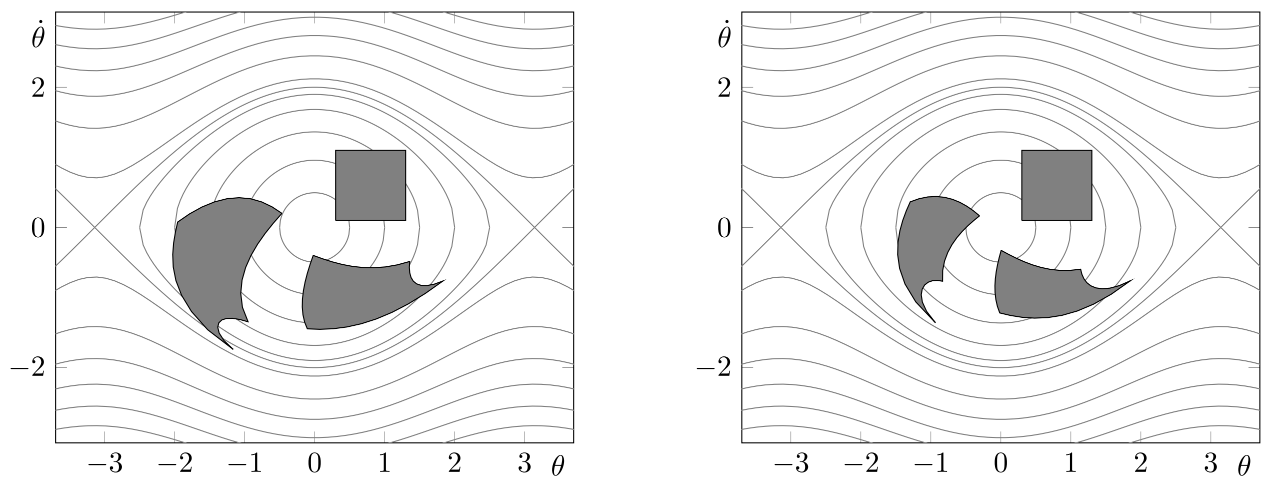


Figure 4 – Transportation of squares by the discrete Lagrangian flow of explicit and symplectic Euler methods.

We shall illustrate here that the symplectic integrators approximatively preserve energy, that is, that the discrete energy is bounded in an constant interval centred around the exact energy value. In the continuous setting, the energy is exactly preserved, and this feature is very important in many applications; the discrete conservation of energy is indeed of primordial importance. This property can be proved in general, but the proof is not developed here; we refer to Hairer [7] for a general exposition. Note that no analogous property in the more general multisymplectic setting has been proved yet, but, as we shall see in part VII, one can reduce the multisymplectic problems to symplectic ones by integrating along the space dimensions of the base space to observe the property of conservation of energy.

The application of the first part of the discrete Euler-Lagrange equations (9.2.1a) in the case of non fixed time steps $(t_i)_{i\in\{0,\dots,N\}}$ leads to

$$\frac{1}{2}\left(\left(\frac{\theta_{k+1}-\theta_k}{t_{k+1}-t_k}\right)^2 - \left(\frac{\theta_k-\theta_{k-1}}{t_k-t_{k-1}}\right)^2\right) - \cos(\theta_k) + \cos(\theta_{k-1}) = 0 \tag{11.0.4}$$

which for $t_{k+1} - t_k = h$ rewrites

$$(\theta_{k+1}-\theta_{k-1})\left(\frac{\theta_{k+1}-2\theta_k+\theta_{k-1}}{h^2}\right) - \cos(\theta_k) + \cos(\theta_{k-1}) = 0 \tag{11.0.5}$$

This is the difference between the discrete energy

$$E_d(t_0, \theta_0, t_1, \theta_1) = \frac{1}{2}\left(\frac{\theta_1 - \theta_0}{t_1 - t_0}\right)^2 - \cos(\theta_0)$$

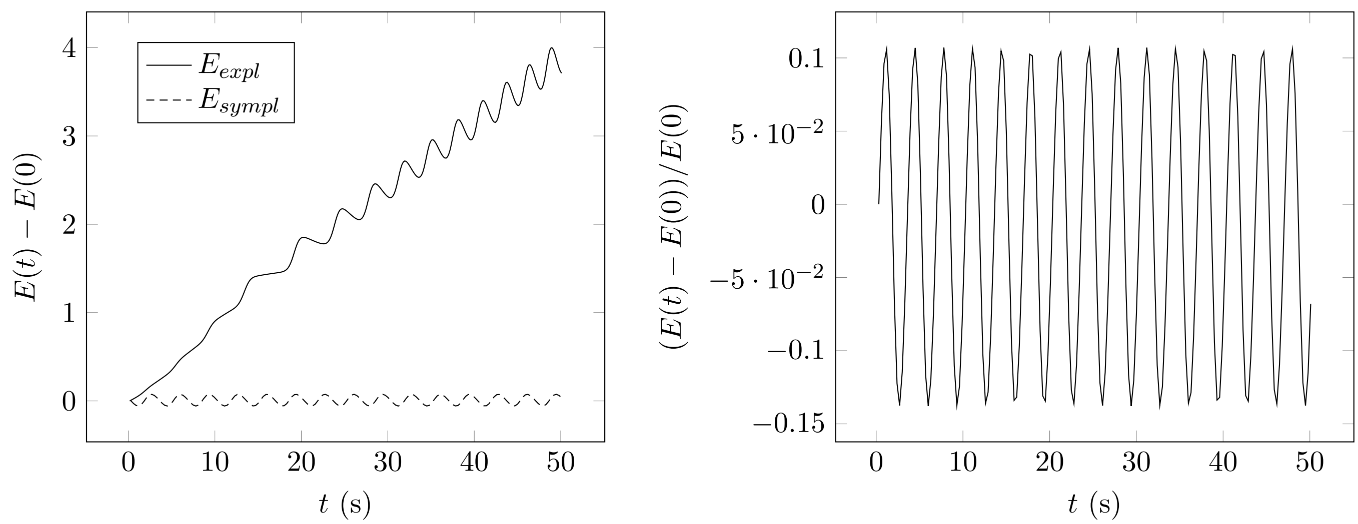
at times t_{k-1} and t_k. We recall the second part of the Euler-Lagrange equations (9.2.1b) that define the numerical scheme (11.0.3), and we replace the approximation of the second derivative of θ_k in (11.0.4) by the term with sine, yielding

$$\frac{\theta_{k+1} - \theta_{k-1}}{2}\sin(\theta_k) = \cos(\theta_k) - \cos(\theta_{k-1}) \tag{11.0.6}$$

Finally, from the Taylor expansion of $\cos(\theta_{k-1})$, we obtain

$$\cos(\theta_{k-1}) = \cos(\theta_k) + (\theta_k - \theta_{k-1})(-\sin(\theta_k)) + \mathcal{O}(h) = \cos(\theta_k) - \left(\frac{\theta_{k+1} - \theta_{k-1}}{2}\right)\sin(\theta_k) + \mathcal{O}(h)$$

proving that equation (11.0.6) holds up to the first order. This yields that the energy is approximately conserved with an order 1 error. An exact energy preserving numerical scheme can be implemented by using adaptive time steps, but is not discussed here; see Hairer [7] for a backward error analysis.

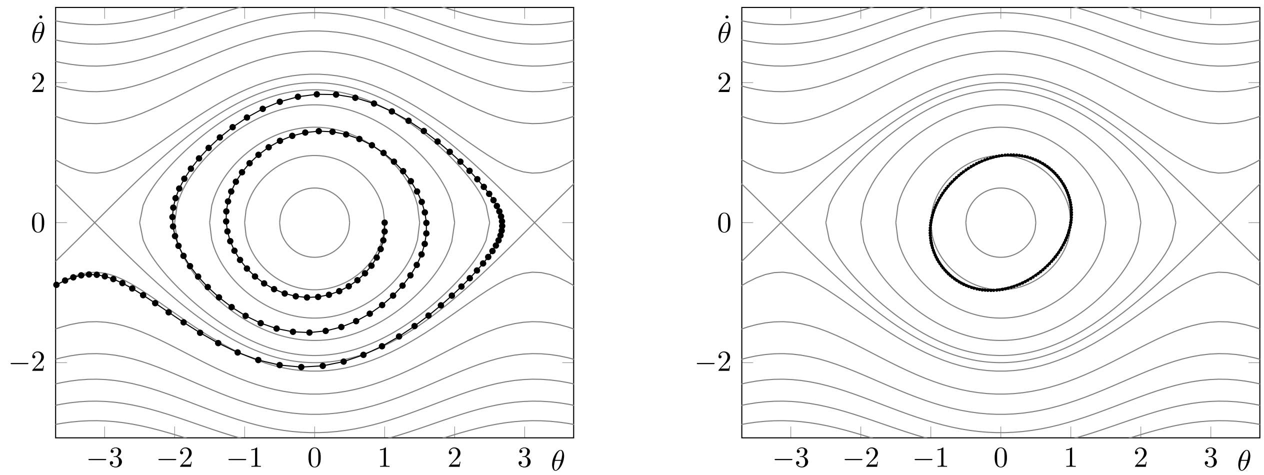


(a) Energy of explicit Euler method with $h = 0.15s$ and symplectic Euler method with $h = 0.3s$.

(b) Energy relative error for symplectic Euler method with $h = 0.3s$.

Figure 5 – Discrete energy for initial condition $(\theta_0, \dot{\theta}_0) = (1, 0)$.

The discrete energy is shown in figure 5 for the pendulum. One can clearly see that the explicit Euler methods artificially "creates" energy and grows as $\mathcal{O}(h)$, whereas the symplectic Euler method oscillates around the true value. In the tangent space, as shown in figure 6, we observe that the explicit Euler method diverges outwards, and the symplectic Euler method evolves on a closed path; the energy stays in a boundary. By noticing the fact that the symplectic methods forms a closed loop in the tangent space in figure 6, and therefore oscillates around a level of energy corresponding to the curves formed by the exact solutions, we get a hint of the link between symplecticity and approximative energy preservation in a more general scope than the way we approached it in our particular case.



(a) Explicit Euler method with $h = 0.15$

(b) Symplectic Euler method with $h = 0.3$

Figure 6 – Discrete approximate solutions of the pendulum problem for initial condition $(\theta_0, \dot{\theta}_0) = (1, 0)$.

Part V
Lagrangian field theory with a Lie group as the fibre

The goal of this part is to follow the same construction as in part III, but in the case of a principle jet bundle with a Lie group as the fibre; a Lie group being a differential manifold, the previous development still holds, but new results arise.

12 Lagrangian field theory

12.1 Jet prolongation and contact form

Let $\pi : M \to E$ be a principal bundle of typical fibre G with G a Lie group. An element of the group is denoted by $g = (y^A)$, and the identity element of G by e. Let $\vec{e}_A = T_e L_g(\vec{\partial}_A)$ be the left invariant basis on TG obtained by left translation of $\vec{\partial}_A$ at e. An element P of E is denoted in local coordinates by (x^μ, y^A), and an element $\bar{\mathrm{P}}$ of J^1E by (x^μ, y^A, ξ^A_μ) where

$$(j^1\phi)^*\xi^A_\mu = \lambda^A\big|_{\mathrm{P}}\left(\frac{\partial \phi^A}{\partial x^\mu}\right)$$

with $\phi \in \Gamma(\pi)$ a section representative of $\bar{\mathrm{P}}$, and λ^A the Maurer-Cartan 1-form defined as the dual basis of the left invariant basis, such that $\lambda^A(\vec{e}_B) = \delta^A_B$. The basis associated to the velocity coordinates ξ^A_μ is denoted by $\vec{\eth}^A_\mu$ and its dual basis by $\mathrm{d}\xi^A_\mu$.

12.1.1 Contact form

The contact form ϑ at any point $\bar{\mathrm{P}}$ is now expressed in local coordinates by $\vartheta_{\bar{\mathrm{P}}} = \vartheta^A \otimes \vec{e}_A$ where $\vartheta^A = \lambda^A - \xi^A_\mu \mathrm{d}x_\mu$. We also need to compute the lift of a vector field, that differs from the standard formalism by a Lie bracket term.

12.1.2 Lift of a vector field

Proposition 6. *Let $Z = \alpha^\mu\vec{\partial}_\mu + \beta^A\vec{e}_A$ be a vector field of $\chi(E)$ with $(\vec{\partial}_\mu, \vec{e}_A, \vec{\eth}^A_\mu)$ the left invariant basis of TJ^1E, the 1-jet prolongation of Z on TJ^1E is the vector field defined at point $\bar{\mathrm{P}} = (x^\mu, y^A, \xi^A_\mu)$ by*

$$j^1Z(\bar{\mathrm{P}}) = \alpha^\mu\vec{\partial}_\mu + \beta^A\vec{e}_A + \left(\frac{\partial \Xi^A}{\partial x^\mu} + \xi^C_\mu T^B_C \frac{\partial \Xi^A}{\partial y^B} + [\xi_\mu, \beta]^A\right)\vec{\eth}^A_\mu$$

where $\Xi^A = j^1Z \lrcorner \vartheta^A = \beta^A - \xi^A_\mu\alpha^\mu$ and $T^B_C = \mathrm{d}y^B(\vec{e}_C)$.

Proof. See Bensoam [1] for a detailed computation of j^1Z. □

12.2 Hamilton principle

12.2.1 Reduced Lagrangian

Let $\omega = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{n+1} \in \Lambda^{n+1}(M)$ be a fixed volume $(n+1)$-form on the manifold M, the *reduced Lagrangian density* is defined as a smooth real-valued function $l \in \mathscr{C}^\infty(J^1E)$, and the associated *reduced Lagrangian form* is defined as the $(n+1)$-form in J^1E

$$\ell = l(x^\mu, y^A, \xi^A_\mu)\omega$$

in a natural local system of coordinates (x^μ, y^A, ξ^A_μ).

12.2.2 Hamilton principle

In this context of a Lie group, the action map $\mathcal{A}$ associated to a Lagrangian system $((E,\pi,M),\ell)$ is defined as

$$\begin{aligned}\mathcal{A} : \Gamma_c(\pi) &\to \mathbb{R}\\ \phi &\mapsto \textstyle\int_{\mathcal{U}} (j^1\phi)^*\ell\end{aligned}$$

Following the same development than the one presented in section 6.3, we compute the variation of action from

$$\delta\mathcal{A} = \int_{\partial\mathcal{U}} (j^1\phi)^*(j^1 Z \lrcorner \ell) + \int_{\mathcal{U}} (j^1\phi)^* j^1 Z \lrcorner \mathrm{d}\ell$$

We obtain

$$\begin{aligned}\delta\mathcal{A} = &\int_{\partial\mathcal{U}} (j^1\phi)^* \left(j^1 Z \lrcorner \ell + \Xi^A \frac{\partial l}{\partial \xi^A_\mu} \mathrm{d}^n x_\mu \right)\\ &- \int_{\mathcal{U}} (j^1\phi)^* \left(\Xi^A \left(\mathrm{d}\left(\frac{\partial l}{\partial \xi^A_\mu} \right) \wedge \mathrm{d}^n x_\mu - \left(\mathrm{ad}^*_{\xi_\nu} \frac{\partial l}{\partial \xi_\nu} \right)_A \omega - T^B_A \frac{\partial l}{\partial y^B}\omega \right) \right)\end{aligned} \tag{12.2.1}$$

where $\Xi = \vartheta^A(j^1 Z) = \beta^A - \xi^A_\mu \alpha^\mu$. The detail of computations is given in appendix B.4.

12.2.3 Euler-Poincaré equations

The Poincaré-Cartan $(n+1)$-form is defined on J^1E by

$$\Theta_\ell = \frac{\partial l}{\partial \xi^A_\mu} \vartheta^A \wedge \mathrm{d}^n x_\mu + \ell$$

After introducing the Euler-Lagrange form Γ_A defined by

$$\Gamma_A = \mathrm{d}\left(\frac{\partial l}{\partial \xi^A_\mu} \right) \wedge \mathrm{d}^n x_\mu - \left(\mathrm{ad}^*_{\xi_\nu} \frac{\partial l}{\partial \xi_\nu} \right)_A \omega - T^B_A \frac{\partial l}{\partial y^B}\omega$$

and noticing that $\Xi^A = j^1 Z \lrcorner \vartheta^A$, the variation of action becomes

$$\delta\mathcal{A} = \int_{\delta\mathcal{U}} (j^1\phi)^*(j^1 Z \lrcorner \Theta_\ell) - \int_{\mathcal{U}} (j^1\phi)^*(j^1 Z \lrcorner (\vartheta^A \wedge \Gamma_A))$$

The Euler-Poincaré equations emerge when taking a vector field Z vanishing on $\partial\mathcal{U}$, giving $\forall A \in \{1,...,N\},\ (j^1\phi)^*\Gamma_A = 0$, that is

$$\left(\frac{\partial}{\partial x_\mu} \frac{\partial l}{\partial \xi^A_\mu} - \left(\mathrm{ad}^*_{\xi_\mu} \frac{\partial l}{\partial \xi_\mu} \right)_A - T^B_A \frac{\partial l}{\partial y^B} \right)(j^1\phi) = 0$$

13 System invariants

13.1 Variation theorem

13.1.1 Lagrangian multisymplectic form

The Lagrangian multisymplectic $(n+2)$-form is computed thanks to the Poincaré-Cartan form as

$$\Omega_\ell = -\mathrm{d}\Theta_\ell = \vartheta^A \wedge \mathrm{d}\left(\frac{\partial l}{\partial \xi^A_\mu} \right) \wedge \mathrm{d}^n x_\mu - \frac{\partial l}{\partial \xi^A_\mu} \mathrm{d}\vartheta^A \wedge \mathrm{d}^n x_\mu - \mathrm{d}l \wedge \omega$$

$$= \vartheta^A \wedge \left(\Gamma_A + \left(\mathrm{ad}^*_{\xi_\mu}\frac{\partial l}{\partial \xi_\mu}\right)_A \omega + T^B_A \frac{\partial l}{\partial y^B}\omega\right)$$
$$+ \frac{\partial l}{\partial \xi^A_\mu}\left(\mathrm{d}\xi^A_\nu \wedge \mathrm{d}x_\nu - \mathrm{d}\lambda^A\right) \wedge \mathrm{d}^n x_\mu - T^B_A \frac{\partial l}{\partial y^B}\mathrm{d}\lambda^A \wedge \omega - \frac{\partial l}{\partial \xi^A_\mu}\mathrm{d}\xi^A_\mu \wedge \omega$$
$$= \vartheta^A \wedge \Gamma_A + \left(\mathrm{ad}^*_{\xi_\mu}\frac{\partial l}{\partial \xi_\mu}\right)_A \lambda^A \wedge \omega + T^B_A \frac{\partial l}{\partial y^B}\lambda^A \wedge \omega$$
$$+ \frac{\partial l}{\partial \xi^A_\mu}\mathrm{d}\xi^A_\mu \wedge \omega - \frac{\partial l}{\partial \xi^A_\mu}\mathrm{d}\lambda^A \wedge \mathrm{d}^n x_\mu - T^B_A \frac{\partial l}{\partial y^B}\lambda^A \wedge \omega - \frac{\partial l}{\partial \xi^A_\mu}\mathrm{d}\xi^A_\mu \wedge \omega$$
$$= \vartheta^A \wedge \Gamma_A + \left(\mathrm{ad}^*_{\xi_\mu}\frac{\partial l}{\partial \xi_\mu}\right)_A \lambda^A \wedge \omega + \frac{\partial l}{\partial \xi^A_\mu}[\lambda, \lambda]^A \wedge \mathrm{d}^n x_\mu$$
$$= \vartheta^A \wedge \Gamma_A + \frac{\partial l}{\partial \xi^A_\mu}\left([\xi_\mu, \lambda]^A \wedge \omega + [\lambda, \lambda]^A \wedge \mathrm{d}^n x_\mu\right)$$

Moreover,

$$[\xi_\mu, \lambda]^A \wedge \omega + [\lambda, \lambda]^A \wedge \mathrm{d}^n x_\mu$$
$$= [\xi_\mu, \vartheta]^A \wedge \omega + [\xi_\mu, \xi_\nu \mathrm{d}x^\nu]^A \wedge \omega + \frac{1}{2}[(\vartheta + \xi_\nu \mathrm{d}x^\nu) \wedge (\vartheta + \xi_\eta \mathrm{d}x^\eta)]^A \wedge \mathrm{d}^n x_\mu$$
$$= [\xi_\mu, \vartheta]^A \wedge \omega + \frac{1}{2}[\vartheta \wedge \vartheta]^A \wedge \mathrm{d}^n x_\mu + [\vartheta \wedge \xi_\nu \mathrm{d}x^\nu]^A \wedge \mathrm{d}^n x_\mu + \frac{1}{2}[\xi_\nu \mathrm{d}x^\nu \wedge \xi_\eta \mathrm{d}x^\eta]^A \wedge \mathrm{d}^n x_\mu$$
$$= [\xi_\mu, \vartheta]^A \wedge \omega + \frac{1}{2}[\vartheta \wedge \vartheta]^A \wedge \mathrm{d}^n x_\mu + [\vartheta, \xi_\mu]^A \wedge \omega + \frac{1}{2}[\xi_\nu \mathrm{d}x^\nu, \xi_\mu]^A \wedge \omega$$
$$= [\vartheta, \vartheta]^A \wedge \mathrm{d}^n x_\mu + \frac{1}{2}[\xi_\nu, \xi_\mu]^A \mathrm{d}x^\nu \wedge \omega = [\vartheta, \vartheta]^A \wedge \mathrm{d}^n x_\mu$$

We finally get

$$\Omega_\ell = \vartheta^A \wedge \Gamma_A + \frac{\partial l}{\partial \xi^A_\mu}[\vartheta, \vartheta]^A \wedge \mathrm{d}^n x_\mu$$

Ω_ℓ vanishes when evaluated on critical sections $(j^1\phi)$. This leads to the variation theorem, which is the same as theorem (1) formulated for reduced Lagrangian systems.

13.1.2 Variation theorem

Theorem 7 (Variation theorem for reduced Lagrangian systems)**.** *Let ϕ be a section of π, the following propositions are equivalent*

(1) *ϕ is a stationary point of the reduced action $\mathcal{A}$*

(2) *ϕ is a solution of the Euler-Poincaré equations*

(3) $\forall W \in \chi(J^1 E),\ (j^1\phi)^*(W \lrcorner \Omega_\ell) = 0$

The proof is essentially the same as the proof of theorem (1) and is therefore not developed here.

13.2 Noether theorem

Since the fibre is a Lie group, invariant Noether current immediately emerge. Let $\Phi_h : G \to G$ with $h \in G$ be a left action of G acting on itself, it leaves the reduced Lagrangian ℓ invariant. From Bensoam [1] we get that the infinitesimal generator S_η of the left action Φ_h is the right invariant vector field X^R_η generated by $\eta \in \mathfrak{g}$. A corresponding left invariant vector field X^L_ν coincides with X^R_η where $\nu = \operatorname{Ad}_{g^{-1}} \eta$. The expression of $j^1 S_\eta$ is obtained from proposition (6)

$$j^1 S_\eta = \left(\operatorname{Ad}_{g^{-1}} \eta\right)^A \vec{e}_A + \left(\frac{\partial \Xi^A}{\partial x^\mu} + \xi^C_\mu T^B_C \frac{\partial \Xi^A}{\partial y^B} + [\xi_\mu, \beta]^A \right) \vec{\eth}^A_\mu$$

where $\Xi^A = j^1 S_\eta \lrcorner \vartheta^A = \left(\operatorname{Ad}_{g^{-1}} \eta\right)^A$. From this we compute the expression of the momentum n-form

$$\begin{aligned} J(S_\eta) &:= (j^1 S_\eta) \lrcorner \Theta_\ell = \frac{\partial l}{\partial \xi^A_\mu} \lambda^A (j^1 S_\eta) \mathrm{d}^n x_\mu - \left(\frac{\partial l}{\partial \xi^A_\mu} \xi^A_\mu - l \right) \cancel{\omega(j^1 S_\eta)} \\ &= \frac{\partial l}{\partial \xi^A_\mu} \left(\operatorname{Ad}_{g^{-1}} \eta\right)^A \mathrm{d}^n x_\mu = \left\langle \frac{\partial l}{\partial \xi_\mu}, \operatorname{Ad}_{g^{-1}} \eta \right\rangle \mathrm{d}^n x_\mu \\ &= \left\langle \operatorname{Ad}^*_{g^{-1}} \pi^\mu, \eta \right\rangle \mathrm{d}^n x_\mu \end{aligned}$$

where $\pi^\mu = \frac{\partial l}{\partial \xi_\mu} \in \mathfrak{g}^*$ is the left momentum. By denoting $\Pi^\mu = \operatorname{Ad}^*_{g^{-1}} \pi^\mu \in \mathfrak{g}^*$ the right momentum associated to π^μ, we define the momentum n-form

$$J := \Pi^\mu \mathrm{d}^n x_\mu$$

where the momentum $J(S_\eta)$ is obtained by $J(S_\eta) = \langle J, \eta \rangle$.

The infinitesimal natural symmetry S_η leaving the reduced Lagrangian ℓ invariant, one may apply the Noether theorem (3), leading to the fact that for any $\eta \in \mathfrak{g}$, the Noether current $(j^1\phi)^* J(S_\eta) = (j^1\phi)^*((j^1 S_\eta) \lrcorner \Theta_{\mathcal{L}})$ is conserved on solutions of the Euler-Poincaré equations. Finally, the invariance of the momentum n-form J on critical sections ϕ of the action is equivalent to an of equation on the right momenta

$$\begin{aligned} \mathrm{d}\left((j^1\phi)^* J\right) &= \frac{\partial \Pi^\mu}{\partial x^\mu} \omega = 0 \\ \Leftrightarrow \sum \frac{\partial \Pi^\mu}{\partial x^\mu} &= 0 \end{aligned}$$

Part VI
Discrete Lagrangian field theory with Lie group as a fibre

This part follows the same development as in part IV in the case where the fibre is a Lie group, in order to adapt the discrete setting to the continuous formulation used in part V

14 Discrete Lagrangian field theory

14.1 Base space discretization

The fibre is now a Lie group G. The implied modifications of the previous results come from the fact that the discrete approximations of the derivatives of the sections with respect to the base space directions can now be expressed with the help of the associated Lie algebra $\mathfrak{g}$, as in the continuous setting of part V. The overall idea is to take advantage of the Lie group structure, and to ensure that it is preserved by the discrete numerical method. This is achieved by updating the group elements by

$$g_{v_j} = g_{v_i}\tau(\Delta_{v_i,v_j}\xi_{v_i,v_j})$$

where $\xi \in \mathfrak{g}$ is an approximation of the left invariant velocity, and $\tau : \mathfrak{g} \to G$ is a local diffeomorphism in the neighbourhood of e_G such that $\tau(0) = e_G$, typically the exponential map.

A way to tackle this problem is to use a multisymplectic generalisation of a Galerkin Lie group variational integrator. We refer to Leok [9] for a presentation of this integrator, and we address the multisymplectic case whitout exhibiting the symplectic case first.

In order to simplify the model and the computations, we make the choice to only consider fixed orthogonal grids as discretization of the base space, that is, the set of vertices V is of the form $\Delta_0\{0,...,K_0\} \times ... \times \Delta_n\{0,...,K_n\}$. The polytopes are chosen to consist of $n+2$ corners, and to be of the form $\sigma_{i_0,...,i_n} = [v_{i_0,...,i_n}, v_{i_0+1,...,i_n}, ..., v_{i_0,...,i_n+1}]$, which in the case of a 2 dimensional base space consists of a set of rectangle triangles, as illustrated in figure 7.

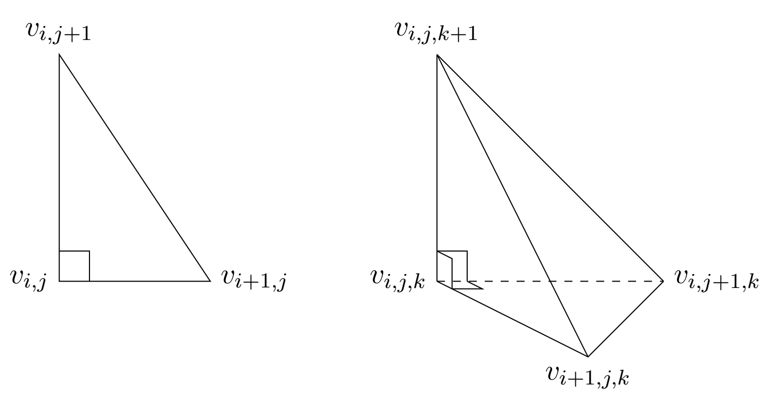


Figure 7 – Two and three dimensional cells.

14.2 Hamilton principle

We introduce the multi indexes $I := (i_0, ..., i_n) \in \{0, ..., K_0 - 1\} \times ... \times \{0, ..., K_n - 1\}$ and, for a given I and $k \in \{0, ..., n\}$, $I_{k+} := (i_0, ..., i_k + 1, ..., i_n)$ and $I_{k-} := (i_0, ..., i_k - 1, ..., i_n)$. We also define

$\sigma_I := [v_{i_0}, ..., v_{i_n}]$, and for each vertex v_I

$$g_I := g(v_I) \in G \tag{14.2.1}$$

$$\xi_I^k := \xi^k(v_I) = \frac{1}{\Delta_k}\tau^{-1}\left(g^{-1}_{i_0,...,i_n} g^{-1}_{i_0,...,i_k+1,i_n}\right)$$
$$= \frac{1}{\Delta_k}\tau^{-1}\left(g_I^{-1} g_{I_{k+}}\right) \in \mathfrak{g} \quad \forall k \in \{0, ..., n\} \tag{14.2.2}$$

A discrete path ψ_d is defined on a polytope σ_I by

$$\psi_d(\sigma_I) = (\sigma_I, g_I, \xi_I^0, \xi_I^n) \in J_d^1 E \tag{14.2.3}$$

where we denote $J_d^1 E := \mathcal{P}_M^{n+2} \times G \times \mathfrak{g}^{n+2}$. Finally we choose the particular definition of the *discrete Lagrangian* as a map $\mathcal{L}_d : P_M^{n+2} \times G \times \mathfrak{g}^{n+1} \to \mathbb{R}$ and we introduce the notation $\mathcal{L}_d^I := \mathcal{L}_d(\sigma_I, g_I, \xi_I^0, ..., \xi_I^n)$.

The *discrete action* $\mathcal{A}_d$ evaluated on a path ψ_d is given by

$$\mathcal{A}_d(\psi_d) = \sum_{\sigma \in \mathcal{P}_M^k} \mathcal{L}_d(\psi_d(\sigma)) \tag{14.2.4}$$

Let $\psi_d \in \mathcal{C}_d$ be a discrete path and $\delta\psi_d \in T_{\psi_d}\mathcal{C}_d$ and arbitrary variation of ψ_d such that $\forall i$, $(\delta\psi_d)_i = (\delta v_i, \delta g_i)$; here we restrain ourself to the case where $\delta v_i = 0$, in other words for a fixed grid, therefore we override the notation in $(\delta\psi_d)_i = \delta g_i$. We compute the variations $\delta\xi_I^k$ as

$$\delta\xi_I^k = \frac{1}{\Delta_k} \mathrm{d}\tau^{-1}_{\Delta_k \xi_I^k} \left(\mathrm{Ad}^*_{\tau(\Delta_k \xi_I^k)}(\zeta_{I_{k+}}) - \zeta_I\right) \tag{14.2.5}$$

where $\zeta_I = g_I^{-1}\delta g_I$. We introduce the discrete momenta $\mu_I^k \in \mathfrak{g}^*$ (see Demoures [4]) by

$$\mu_I^k := \left(\mathrm{d}\tau^{-1}_{\Delta_k \xi_I}{}^*\right) D_{\xi^k} \mathcal{L}_d^I \tag{14.2.6}$$

The variation of action at ψ_d in direction $\delta\psi_d$ is then expressed by

$$\begin{aligned}
\delta\mathcal{A}_d(\psi_d) &= \mathrm{d}\mathcal{A}_d(\psi_d) \cdot \delta\psi_d = \sum_{\sigma \in \mathcal{P}_M^k} \mathrm{d}\mathcal{L}_d(\sigma, g_\sigma, \xi_\sigma^0, ..., \xi_\sigma^n) \cdot \delta\psi_d \\
&= \sum_I \mathrm{d}\mathcal{L}_d(\sigma_I, g_I, \xi_I^0, ..., \xi_I^n) \cdot \delta\psi_d = \sum_I \left(D_g \mathcal{L}_d^I \cdot \delta g_I + \sum_{k=0}^n D_{\xi^k} \mathcal{L}_d^I \cdot \delta\xi_I^k\right) \\
&= \sum_I \left(D_g \mathcal{L}_d^I \cdot \delta g_I + \sum_{k=0}^n \left\langle D_{\xi^k}\mathcal{L}_d^I, \frac{1}{\Delta_k}\mathrm{d}\tau^{-1}_{\Delta_k \xi_I^k}\left(-\zeta_I + \mathrm{Ad}_{\tau(\Delta_k \xi_I^k)}\zeta_{I_{k+}}\right)\right\rangle\right) \\
&= \sum_I \left(D_g \mathcal{L}_d^I \cdot \delta g_I + \sum_{k=0}^n \left\langle -\frac{1}{\Delta_k}\left(\mathrm{d}\tau^{-1}_{\Delta_k \xi_I^k}{}^*\right) D_{\xi_I^k}\mathcal{L}_d^I, \zeta_I\right\rangle \right. \\
&\qquad\qquad \left. + \left\langle \frac{1}{\Delta_k}\mathrm{Ad}^*_{\tau(\Delta_k \xi_I^k)}\left(\mathrm{d}\tau^{-1}_{\Delta_k \xi_I^k}{}^*\right) D_{\xi_I^k}\mathcal{L}_d^I, \zeta_{I_{k+}}\right\rangle\right) \\
&= \sum_I \left((g_I)^{-1} D_g \mathcal{L}_d^I \cdot \zeta_I + \sum_{k=0}^n -\frac{1}{\Delta_k}\mu_I^k \cdot \zeta_I + \frac{1}{\Delta_k}\mathrm{Ad}^*_{\tau(\Delta_k \xi_I^k)}\mu_I^k \cdot \zeta_{I_{k+}}\right)
\end{aligned} \tag{14.2.7}$$

Introducing the notation

$$a_I = (g_I)^{-1} D_g \ell_d^I + \sum_{k=0}^n -\frac{1}{\Delta_k}\mu_I^k \qquad \text{and} \qquad b_I^k = \frac{1}{\Delta_k}\mathrm{Ad}^*_{\tau(\Delta_k \xi^k_{I_{k-}})}\mu^k_{I_{k-}}$$

we obtain for the variation of action

$$\begin{aligned}\delta\mathcal{A}_d(\psi_d) &= \sum_{i_0=0}^{K_0-1}\dots\sum_{i_n=0}^{K_n-1} a_I\cdot\zeta_I + \sum_{i_0=0}^{K_0-1}\dots\sum_{i_n=0}^{K_n-1}\sum_{k=0}^{n} b^k_{I_{k+}}\cdot\zeta_{I_{k+}}\\ &= \sum_{i_0=0}^{K_0-1}\dots\sum_{i_n=0}^{K_n-1} a_I\cdot\zeta_I + \sum_{i_0=0}^{K_0-1}\dots\sum_{i_k=1}^{K_k}\dots\sum_{i_n=0}^{K_n-1}\sum_{k=0}^{n} b^k_I\cdot\zeta_I\\ &= \sum_{i_0=1}^{K_0-1}\dots\sum_{i_n=1}^{K_n-1}\left(a_I+\sum_{k=0}^{n} b^k_I\right)\cdot\zeta_I + A\end{aligned}$$

where A are the terms on the border of the discrete domain. With the current notation and summation conventions, which has been chosen for its convenient way to express ξ^k_I, A is artificially complex and very painful to explicit because of the indices combinations. This is why we express in the following the terms on the boundary with respect to sums over σ.

We define $D_{DEL}\mathcal{L}^I_d$ by

$$D_{DEL}\mathcal{L}_d(\sigma, g_\sigma, \xi^1_\sigma, ..., \xi^k_\sigma) = \left\langle (g_I)^{-1}D_g\mathcal{L}^I_d + \sum_{k=0}^{n}\frac{1}{\Delta_k}\left(-\mu^k_I + \mathrm{Ad}^*_{\tau(\Delta_k\xi^k_{I_{k-}})}\,\mu^k_I\right), g_I^{-1}\mathrm{d}g_I\right\rangle \tag{14.2.8}$$

where we use the short cut notation $\mathrm{d}\zeta_I = g_I^{-1}\mathrm{d}g_I$, and the discrete Cartan forms $\Theta^i_{\mathcal{L}_d}$ by

$$\Theta^1_{\mathcal{L}_d}(\sigma, g_\sigma, \xi^0_\sigma, ..., \xi^n_\sigma) = \left\langle g_\sigma^{-1}D_g\mathcal{L}^I_d - \sum_{k=0}^{n}\frac{1}{\Delta_k}\mu^k_I, g_I^{-1}\mathrm{d}g_I\right\rangle \tag{14.2.9}$$

$$\Theta^{k+2}_{\mathcal{L}_d}(\sigma, g_\sigma, \xi^0_\sigma, ..., \xi^n_\sigma) = \left\langle \frac{1}{\Delta_k}\mathrm{Ad}^*_{\tau(\Delta_k\xi^k_I)}\,\mu^k_I, g^{-1}_{I_{k+}}\mathrm{d}g_{I_{k+}}\right\rangle \tag{14.2.10}$$

Taking vanishing variations on the border of the domain yields the discrete Euler-Poincaré equations, given for any $n+1$ dimension of the base space at $I = (i_0, ..., i_n) \in \{1, ..., K_0-1\}\times ... \times\{1, ..., K_n-1\}$ by

$$(g_I)^{-1}D_g\mathcal{L}^I_d + \sum_{k=0}^{n}\frac{1}{\Delta_k}\left(-\mu^k_I + \mathrm{Ad}^*_{\tau(\Delta_k\xi^k_{I_{k-}})}\,\mu^k_I\right) = 0 \tag{14.2.11}$$

15 System invariants

15.1 Lagrangian symmetries and Noether's theorem

Let Φ_h with $h \in G$ be a left action $\Phi_h : G \to G$ of G acting on itself, the induced action on J^1_dE is denoted $\Phi^{J^1_dE}_g$ and the associated infinitesimal generator $\eta^{J^1_dE}$. In that case the discrete Lagrangian $\mathcal{L}_d$ is invariant with respect to the action $\Phi^{J^1_dE}$, and is therefore infinitesimally invariant with respect to $\eta^{J^1_dE}$.

The discrete momentum maps $J^i_{\mathcal{L}_d} : J^1_dE \to \mathfrak{g}^*$, $i \in \{0, ..., n\}$ are defined for all $\eta \in \mathfrak{g}$ by

$$\left\langle J^i_{\mathcal{L}_d}, \eta\right\rangle := \eta^{J^1_dE} \lrcorner \Theta^i_{\mathcal{L}_d}$$

The variations of the path induced by the infinitesimal generator of the left action is $\delta g_I = \eta_I g_I$. Given the discrete Cartan forms expressions (14.2.9) and (14.2.10), the momentum maps can be computed for any multi index I and $\eta \in \mathfrak{g}$ as

$$\left\langle J^1_{\mathcal{L}_d}(\sigma_I, g_I, \xi^0_I, ..., \xi^n_I), \eta\right\rangle = \left\langle g_\sigma^{-1}D_g\mathcal{L}^I_d - \sum_{k=0}^{n}\frac{1}{\Delta_k}\mu^k_I, g_I^{-1}\eta_I g_I\right\rangle$$

$$= \left\langle g_\sigma^{-1} D_g \mathcal{L}_d^I - \sum_{k=0}^{n} \frac{1}{\Delta_k} \mu_I^k, \mathrm{Ad}_{g_I^{-1}}\, \eta_I \right\rangle$$

$$= \left\langle \mathrm{Ad}^*_{g_I^{-1}} \left(g_\sigma^{-1} D_g \mathcal{L}_d^I - \sum_{k=0}^{n} \frac{1}{\Delta_k} \mu_I^k \right), \eta_I \right\rangle$$

$$\left\langle J^{k+2}_{\mathcal{L}_d}(\sigma_I, g_I, \xi_I^0, ..., \xi_I^n), \eta \right\rangle = \left\langle \frac{1}{\Delta_k}\, \mathrm{Ad}^*_{\tau(\Delta_k \xi_I^k)}\, \mu_I^k, \mathrm{Ad}_{g^{-1}_{I_{k+}}}\, \eta_{I_{k+}} \right\rangle$$

$$= \left\langle \frac{1}{\Delta_k}\, \mathrm{Ad}^*_{g^{-1}_{I_{k+}}} \left(\mathrm{Ad}^*_{\tau(\Delta_k \xi_I^k)}\, \mu_I^k \right), \eta_{I_{k+}} \right\rangle$$

We then identify

$$J^1_{\mathcal{L}_d}(\sigma_I, g_I, \xi_I^0, ..., \xi_I^n) = \mathrm{Ad}^*_{g_I^{-1}} \left(g_\sigma^{-1} D_g \mathcal{L}_d^I - \sum_{k=0}^{n} \frac{1}{\Delta_k} \mu_I^k \right) \tag{15.1.1}$$

$$J^{k+2}_{\mathcal{L}_d}(\sigma_I, g_I, \xi_I^0, ..., \xi_I^n) = \frac{1}{\Delta_k}\, \mathrm{Ad}^*_{g_I^{-1}}\, \mu_I^k \tag{15.1.2}$$

where we used the fact that

$$\mathrm{Ad}^*_{g^{-1}_{I_{k+}}}\, \mathrm{Ad}^*_{\tau(\Delta_k \xi_I^k)} = \mathrm{Ad}^*_{g^{-1}_{I_{k+}}}\, \mathrm{Ad}^*_{g_I^{-1} g_{I_{k+}}} = \mathrm{Ad}^*_{g_I^{-1}}$$

Evaluating the sum the momentum maps against $\eta \in \mathfrak{g}$ gives us

$$\left\langle \sum_{i=1}^{n+2} J^i_{\mathcal{L}_d}(\sigma_I, g_I, \xi_I^0, ..., \xi_I^n), \eta \right\rangle = \left\langle \mathrm{Ad}^*_{g_I^{-1}}\, g_\sigma^{-1} D_g \mathcal{L}_d^I, \eta_I \right\rangle = \left\langle g_\sigma^{-1} D_g \mathcal{L}_d^I, \mathrm{Ad}_{g_I^{-1}}\, \eta_I \right\rangle$$

$$= \left\langle D_g \mathcal{L}_d^I, \eta_I g_I \right\rangle = 0$$

since $\mathcal{L}_d$ is infinitesimally invariant with respect to η. For any path ψ_d solution of the discrete Euler-Lagrange equations and $\sigma \in \mathcal{P}_M^{n+2}$, we obtain the statement of the local Noether theorem

$$\sum_{i=1}^{n+2} J^i_{\mathcal{L}_d}(\psi_d(\sigma)) = 0 \tag{15.1.3}$$

The global Noether theorem (6) still holds, only the expressions of the momentum maps differ.

Part VII
Reissner beam

The Reissner beam models a non linear system on a 2 dimensional base space. It is a system of particular interest to model the non linear behaviour of strings when they are subject to large displacements, since in that case the hypothesis of the perfect string do not hold any more.

Since its configuration space, the 6 dimensional space of translations and rotations $SE(3)$, is a Lie group, it can be handled by the formalism presented in part V. This model is presented in section 16, followed by a short investigation of possible integrable solutions in section 17. A discretization of the model is then introduced in section 18 and applied to obtain numerical results. Finally, applications for sound synthesis are presented in section 19.

We refer to appendix A for a presentation of the mathematical objects and operations involved in the use of Lie group $SE(3)$ on which the model of the Reissner beam is based.

16 Nonlinear model

Let us consider a beam of length L with constant circular cross-section of area A and radius a, and density ρ, each section of the beam is considered to be a rigid body evolving in the three-dimensional space, thus having six degrees of liberty. The beam configuration is entirely described by the knowledge, for any couple $(t, s) \in M = \mathbb{R} \times [0, L]$ consisting of the time date and the curvilinear position, of the translation $r(t, s) \in \mathbb{R}^3$ and rotation $R(t, s) \in SO(3)$ of each section with respect to a reference configuration Σ_0.

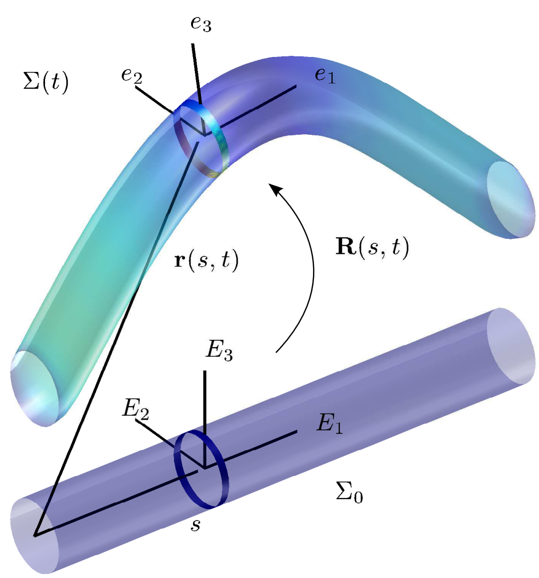


Figure 8 – $\Sigma(t)$ and Σ_0, respectively current and reference configurations of the beam.

Let E_1 be a unit vector along the axis of the reference beam, oriented by the increasing s coordinate, and E_2, E_3 any pair of unit vectors such that (E_1, E_2, E_3) is a directly oriented orthonormal basis of $\mathbb{R}^3$. As represented in figure 8, the coordinates at time t of any point of the section located at curvilinear position s is

$$x(t, s) = r(t, s) + R(t, s)X$$

where $X = \alpha E_2 + \beta E_3$ with $\alpha^2 + \beta^2 \leq a^2$ is the position within the section of the beam at s. The previous equality can be expressed thanks to a unique map $H(t,s) \in SE(3) \simeq SO(3) \times \mathbb{R}^3$ such that

$$y(t,s) := \begin{bmatrix} x(t,s) \\ 1 \end{bmatrix} = \begin{bmatrix} R(t,s) & r(t,s) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix} = \hat{H}(t,s)Y$$

$H(t,s)$ is an element of the Lie group $SE(3) = G$ (see appendix A for a detailed explanation of the notation of $SE(3)$ elements). To any motion of the beam corresponds a section ϕ such that $\forall(t,s),\ \phi(t,s) = (t,s,H(t,s))$.

We define (χ_L, ϵ_L) the left invariant lift of the tangent vectors $\dot{H} := \frac{\partial H}{\partial t}$ and $H' := \frac{\partial H}{\partial s}$ to the Lie algebra $\mathfrak{g} = \mathfrak{se}(3)$ by

$$\chi_L^A = \lambda^A\big|_H \left(\dot{H}\right), \quad \epsilon_L^A = \lambda^A\big|_H \left(H'\right)$$

In the matrix representation, we have the simple relation $\hat{\xi}_\mu = \hat{H}^{-1}\frac{\partial \hat{H}}{\partial x_\mu}$. We get

$$\hat{\chi}_L = \hat{H}^{-1}\dot{\hat{H}} = \begin{bmatrix} R^T\dot{R} & R^T\dot{r} \\ 0 & 0 \end{bmatrix} := \begin{bmatrix} \hat{\omega}_L & \gamma_L \\ 0 & 0 \end{bmatrix} \tag{16.0.4}$$

$$\hat{\epsilon}_L = \hat{H}^{-1}\hat{H}' = \begin{bmatrix} R^TR' & R^Tr' \\ 0 & 0 \end{bmatrix} := \begin{bmatrix} \hat{\Omega}_L & \Gamma_L \\ 0 & 0 \end{bmatrix} \tag{16.0.5}$$

From now on, we override the notation $\chi_: = \chi_L$ and $\epsilon := \epsilon_L$.

To sum up, the base space $M = \mathbb{R} \times [0,L]$ consists of a time and spatial coordinates, the configuration space $G = SE(3)$ is a Lie group, hence the motion of the beam is represented by a section of the total space $J^1E := M \times G \times \mathfrak{g}^2$ given for any $(t,s) \in M$ by $j^1\phi(t,s) = (t,s,H(t,s),\chi(t,s),\epsilon(t,s))$.

Let the reduced Lagrangian be defined as the difference between kinetic and potential energy $l = K - U$. We compute for the kinetic energy

$$\begin{aligned} K(H,\chi,\epsilon) &= \iint_S \frac{1}{2}\rho\dot{x}^T\dot{x}dSds = \iint_S \frac{1}{2}\rho\dot{y}^T\dot{y}dS \\ &= \iint_S \frac{1}{2}\rho Y^T\dot{\hat{H}}^T\dot{\hat{H}}YdSds = \iint_S \frac{1}{2}\rho\left(X^T\dot{R}^T\dot{R}X + 2\dot{r}^T\dot{R}X + \dot{r}^T\dot{r}\right)dS \\ &= \frac{1}{2}\left(\iint_S \rho\left(\gamma^TR^TR\gamma + \omega^T\hat{X}^TR^TR\hat{X}\omega\right)dS + 2\dot{r}^T\dot{R}\underbrace{\iint_S \rho X dS}_{=0}\right) \\ &= \frac{1}{2}\iint_S \rho\left(\gamma^T\gamma + \omega^T\hat{X}^T\hat{X}\omega\right)dS \\ &= \frac{1}{2}\chi^T\mathbb{J}\chi \end{aligned}$$

where the tensor of inertia $\mathbb{J}$ is expressed by

$$\mathbb{J} = \iint_S \rho\begin{bmatrix} \hat{X}^T\hat{X} & 0 \\ 0 & \mathbb{I}_3 \end{bmatrix}dS = \begin{bmatrix} \mathbb{J}_r & 0 \\ 0 & \mathbb{J}_d \end{bmatrix}, \quad \mathbb{J}_r = \begin{bmatrix} \rho I_\rho & 0 & 0 \\ 0 & \rho I_a & 0 \\ 0 & 0 & \rho I_a \end{bmatrix}, \quad \mathbb{J}_d = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \tag{16.0.6}$$

$$I_\rho = \iint_S (Y^2 + Z^2)dS = \pi a^2, \qquad I_a = \iint_S Y^2dS = \iint_S Z^2dS = \frac{\pi a^2}{2}$$

and $m = \rho A$ is the linear mass.

In the same manner we have

$$\begin{aligned} U(H,\chi,\epsilon) &= \frac{1}{2}\left((\Omega-\Omega^0)^T M + (\Gamma-\Gamma^0)^T F\right) \\ &= \frac{1}{2}\left((\Omega-\Omega^0)^T \mathbb{C}_r(\Omega-\Omega^0) + (\Gamma-\Gamma^0)^T \mathbb{C}_d(\Gamma-\Gamma^0)\right) \\ &= \frac{1}{2}\left((\epsilon-\epsilon^0)^T \mathbb{C}(\epsilon-\epsilon^0)\right) \end{aligned}$$

where $F = \mathbb{C}_d(\Gamma-\Gamma^0)$ and $M = \mathbb{C}_r(\Omega-\Omega^0)$ are respectively the internal force and torque, and the Hook tensor $\mathbb{C}$ is expressed by $\mathbb{C} = \begin{bmatrix} \mathbb{C}_r & 0 \\ 0 & \mathbb{C}_d \end{bmatrix}$ where

$$\mathbb{C}_r = \begin{bmatrix} GI_\rho & 0 & 0 \\ 0 & EI_a & 0 \\ 0 & 0 & EI_a \end{bmatrix}, \quad \mathbb{C}_d = \begin{bmatrix} EA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & GA \end{bmatrix} \tag{16.0.7}$$

with E, G and A respectively the Young modulus, the shear coefficient and the cross-sectional area, and $\epsilon^0 = (\Omega^0, \Gamma^0)$ is the deformation of the reference configuration such that $\epsilon^0 = E_4 = (0,0,0,1,0,0)$.

Finally, we get the left invariant Lagrangian density l on the natural coordinate system $(t, s, H, \chi, \epsilon)$ and its associated reduced Lagrangian form ℓ

$$\begin{aligned} l(\chi,\epsilon) &= \frac{1}{2}\left(\chi^T \mathbb{J}\chi - (\epsilon-\epsilon^0)^T \mathbb{C}(\epsilon-\epsilon^0)\right) \\ \ell(\chi,\epsilon) &= l(\chi,\epsilon)\omega \end{aligned}$$

where $\omega = dt \wedge ds$.

Applying the Hamilton principle gives us the Euler-Poincaré equations $\forall A \in \{1, ..., 6\}$

$$\frac{\partial}{\partial t}\frac{\partial l}{\partial \chi^A} + \frac{\partial}{\partial s}\frac{\partial l}{\partial \epsilon^A} - \left(\mathrm{ad}^*_\chi \frac{\partial l}{\partial \chi}\right)_A - \left(\mathrm{ad}^*_\epsilon \frac{\partial l}{\partial \epsilon}\right)_A = 0. \tag{16.0.8}$$

After computing the partial derivatives of l and defining the left momenta

$$\pi := \frac{\partial l}{\partial \chi} = \mathbb{J}\chi \qquad \text{and} \qquad \sigma := \frac{\partial l}{\partial \epsilon} = -\mathbb{C}(\epsilon-\epsilon^0)$$

in $\mathfrak{se}(3)^*$, we get from equation (16.0.8) the equivalent formulation

$$\frac{\partial}{\partial t}\pi + \frac{\partial}{\partial s}\sigma - \mathrm{ad}^*_\chi \pi - \mathrm{ad}^*_\epsilon \sigma = 0. \tag{16.0.9}$$

Since the fibre $SO(3)$ is a Lie group, the Noether theorem (3) applies. We define the right momenta Π and Σ presented in section 15, associated respectively to π and σ as

$$\Pi = \mathrm{Ad}^*_{H^{-1}} \pi \qquad \text{and} \qquad \Sigma = \mathrm{Ad}^*_{H^{-1}} \sigma$$

We introduce the momentum 1-form $J = \Pi \mathrm{d}s - \Sigma \mathrm{d}t$, and we recall from section 15 that the conservation of the Noether current $(j^1\phi)^* J$ is equivalent the following equation on the right momenta

$$\frac{\partial}{\partial t}\Pi + \frac{\partial}{\partial s}\Sigma = 0 \tag{16.0.10}$$

This conservation law can also be retrieved from (16.0.9) by introducing introduce the right invariant tangent vector lifts χ_R and ϵ_R as

$$\chi_R = \dot{H}H^{-1} = \mathrm{Ad}_H \chi \qquad \text{and} \qquad \epsilon_R = H'H^{-1} = \mathrm{Ad}_H \epsilon$$

in which case we have $\Pi = \frac{\partial l}{\partial \chi_R}$ and $\Sigma = \frac{\partial l}{\partial \epsilon_R}$. Equations (16.0.9) and (16.0.10), respectively in the left and right representation, are indeed equivalent.

In order to obtain a well-posed problem, we compute the compatibility condition, obtained by differentiating (16.0.4) and (16.0.5)

$$\begin{aligned}\frac{\partial}{\partial s}\hat{\chi} - \frac{\partial}{\partial t}\hat{\epsilon} &= \frac{\partial}{\partial s}(\hat{H}^{-1})\frac{\partial}{\partial t}\hat{H} - \frac{\partial}{\partial t}(\hat{H}^{-1})\frac{\partial}{\partial s}\hat{H} + H^{-1}\left(\frac{\partial^2}{\partial s \partial t}\hat{H} - \frac{\partial^2}{\partial t}\partial s \hat{H}\right) \\ &= \frac{\partial}{\partial s}(\hat{H}^{-1})\hat{H}\hat{\chi} - \frac{\partial}{\partial t}(\hat{H}^{-1})\hat{H}\hat{\epsilon} = \hat{\epsilon}^T\hat{\chi} - \hat{\chi}^T\hat{\epsilon} = \hat{\chi}\hat{\epsilon} - \hat{\epsilon}\hat{\chi} = [\hat{\chi}, \hat{\epsilon}]\end{aligned}$$

yielding

$$\frac{\partial}{\partial s}\chi - \frac{\partial}{\partial t}\epsilon = [\chi, \epsilon] \qquad (16.0.11)$$

In the right representation, equation (16.0.11) becomes

$$\frac{\partial}{\partial s}\chi_R - \frac{\partial}{\partial t}\epsilon_R = [\chi_R, \epsilon_R] \qquad (16.0.12)$$

The non-linear model for the Reissner beam is given by equations (16.0.9) and (16.0.11), or equivalently by (16.0.10) and (16.0.12).

17 Integrable solutions

Under some assumptions, the previous system is completely integrable. The overall idea of the resolution has been developed with Hélein and Bensoam in [2], and is reproduced here. However, it has not led to a formal resolution yet, this is why no explicit solution is given.

The goal is to turn the problem into finding the connection maps solutions of a zero equation. By doing so, the problem becomes equivalent to a principal chiral model, for which explicit integrable solutions exist. The study of this category of problems is given for example in Mañas [10].

To reformulate the problem in an appropriate manner, we use the right representation of the non-linear model. We need the equations (16.0.12) and (16.0.10) to linearly depend on χ_R and ϵ_R, hence Π and χ_R (respectively Σ and ϵ_R) must verify a linear relation independent of H.

We make the hypothesis that the tensors $\mathbb{J}$ and $\mathbb{C}$ are proportional to the identity, and we write $\mathbb{J} = J\mathbb{I}$ and $\mathbb{C} = C\mathbb{I}$; in that case, the Ad operator and the matrix product commute, and the relation between Π and χ_R simply becomes $\Pi = \mathrm{Ad}^*_{H^{-1}}(\mathbb{J}\,\mathrm{Ad}_{H^{-1}}\,\chi_R) = J\chi_R$. We chose to take J and C equal to one, and $\epsilon_R^0 = 0$. The equations (16.0.12) and (16.0.10) respectively become :

$$\begin{cases} \frac{\partial}{\partial s}\chi_R - \frac{\partial}{\partial t}\epsilon_R - [\epsilon_R, \chi_R] = 0 \\ \frac{\partial}{\partial t}\chi_R - \frac{\partial}{\partial s}\epsilon_R = 0 \end{cases}$$

The hypothesis that have been made do not correspond to realistic physical properties, and cannot be used as is to compare experimental results and formal resolutions, but are nonetheless desirable to quantify the numerical results in a particular case. Moreover, a future work might show that the hypothesis could be weakened, in particular on the hypothesis of proportionality of the tensors to the identity.

Consider the 1-form $\omega = \chi_R \mathrm{d}t + \epsilon_R \mathrm{d}s$ and its Hodge star $\star\omega = \chi_R \mathrm{d}s + \epsilon_R \mathrm{d}t$, the previous system is equivalent to

$$\begin{cases} \mathrm{d}\omega - \omega \wedge \omega = 0 \\ \mathrm{d}(\star\omega) = 0 \end{cases} \qquad (17.0.13)$$

We introduce $\omega_L = \frac{1}{2}(\omega + \star\omega) = \frac{1}{2}(\epsilon_s + \chi_s)\mathrm{d}(s+t)$ and $\omega_R = \frac{1}{2}(\omega - \star\omega) = \frac{1}{2}(\epsilon_s - \chi_s)\mathrm{d}(s-t)$, where L and R respectively stands for left and right moving. Those forms check $\star\omega_L = \omega_L$ and $\star\omega_R = -\omega_R$.

Thanks to the *spectral parameter* $\lambda \in (\mathbb{C} \cup \{\infty\}) \backslash \{\pm 1\} := \mathcal{D}$, we introduce the family of connection forms $(\omega_\lambda)_{\lambda \in \mathcal{D}}$ defined by

$$\omega_\lambda = \frac{\omega_L}{1+\lambda} + \frac{\omega_R}{1-\lambda} = \frac{\omega - \lambda \star \omega}{1-\lambda^2}.$$

We have $\omega_\lambda = \omega$ when $\lambda = 0$. Moreover, system (17.0.13) is equivalent $\forall \lambda \in \mathcal{D}$ to the equation

$$d\omega_\lambda - \omega_\lambda \wedge \omega_\lambda = 0.$$

This equation is a necessary and sufficient condition for the existence of a family of maps $(H_\lambda)_{\lambda \in \mathcal{D}}$ from $\mathbb{R}^2$ to $SE(3)^{\mathbb{C}}$ the complexification of $SE(3)$ such that

$$dH_\lambda = \omega_\lambda H_\lambda$$

on $\mathbb{R}^2$.

There exist a correspondence between the family $(H_\lambda)_{\lambda \in \mathcal{D}}$ and the solutions of linear wave equations through the transformations called the *dressing* and *undressing procedures*, which are described in Hélein [2]. In particular, the correspondence could be made with the soliton solutions, and further investigations could focus on the existence of such solutions in the case of weaker hypothesis on the tensors $\mathbb{J}$ and $\mathbb{C}$.

18 Numerical solutions

In this section, we present a discretization of the Reissner beam model with the help of the formalism developed in part VI. The discrete equations are then used to derive numerical methods, and are applied on various cases.

18.1 Discretization

Let us consider a set $\mathcal{U} = [0,T] \times [0,L]$ on the base space, let $h, l \in \mathbb{R}^2$ and $N, P \in \mathbb{N}^2$ be such that $hN = T$ and $lP = L$, we choose a fixed set of vertices

$$V = \left\{ v_{i+j(N+1)} := v_i^j = (i,j) \;/\; (i,j) \in \{0,...,N\} \times \{0,...,P\} \right\}$$

and define $\forall i \in \{0,...,N\}$, $t_i = hi$ and $\forall j \in \{0,...,P\}$, $s_j = lj$. The mesh consists of a set of triangles $\mathcal{P}_M^3 := \left\{ \sigma_i^j = [v_i^j, v_{i+1}^j, v_i^{j+1}] \;/\; (i,j) \in \{0,...,N-1\} \times \{0,...,P-1\} \right\}$. An illustration of such a type of mesh is given in figure 9.

Lie algebra elements χ and ϵ in $\mathfrak{se}(3)$ will be represented with the use of a local diffeomorphism $\tau : \mathfrak{se}(3) \to SE(3)$ around the origin verifying $\tau(0) = e$ such that $\tau(h\chi_i^j) = (H_i^j)^{-1} H_{i+1}^j$ and $\tau(l\epsilon_i^j) = (H_i^j)^{-1} H_i^{j+1}$ as presented in section 14. This yields for all $(i,j) \in \{0,...,N-1\} \times \{0,...,P-1\}$

$$\chi_i^j = \tau^{-1}\left((H_i^j)^{-1} H_{i+1}^j \right)/h \qquad \text{and} \qquad \epsilon_i^j = \tau^{-1}\left((H_i^j)^{-1} H_i^{j+1} \right)/l \tag{18.1.1}$$

There are numerous possibilities for the definition of τ; as proposed by Leok [9], we choose to use the analogous of the Cayley map for $SE(3)$, which gives a good approximation of the exponential map for small displacements around the origin. This choice of approximation is motivated by the fact that the two maps are close near the origin, and the Cayley map is easier to invert and faster to compute in the case of the Lie group $SE(3)$, thus interesting for numerical applications. The following results regarding the Cayley map are developed in Demoures [3].

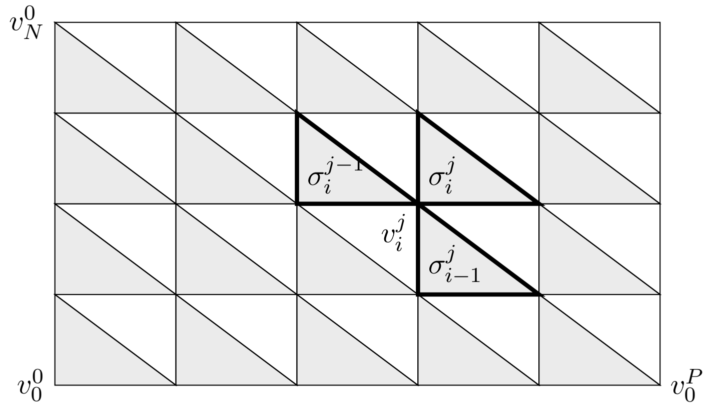


Figure 9 – Base space discretization.

The Cayley map for the rotation group $SO(3)$ is an isomorphism between $\mathfrak{so}(3)$ and $SO(3)$, and is defined for any $\omega \in \mathfrak{so}(3)$ by

$$\operatorname{cay}(\omega) = \left(I_3 - \frac{\hat{\omega}}{2}\right)^{-1}\left(I_3 + \frac{\hat{\omega}}{2}\right) = I_3 + \frac{4}{4+\omega^T\omega}\left(\hat{\omega} + \frac{\hat{\omega}^2}{2}\right).$$

The Cayley map is invertible for any rotation $R \in SO(3)$ which is not a rotation of angle $\pm\pi$, in which case $tr(R) = -1$, and its inverse is given by

$$\operatorname{cay}^{-1}(R) = 2\left(R - I_3\right)\left(R + I_3\right)^{-1} = \frac{2}{1+tr(R)}\left(R - R^T\right).$$

The relation between the Cayley map and the exponential map is found by using the Rodrigues formula for the exponential of an element $\omega \in \mathfrak{so}(3)$ and yields

$$\operatorname{cay}^{-1}(e^{\hat{\omega}}) = \frac{\tan\left(\frac{\|\omega\|}{2}\right)}{\frac{\|\omega\|}{2}}\hat{\omega}$$

which is equivalent to $\hat{\omega}$ when $\omega \to 0$, which shows that the Cayley and exponential maps are close in a neighbourhood of the origin.

The Cayley map can be extended for the $SE(3)$ group as the map τ defined for any $\chi = (\omega, \gamma) \in \mathfrak{se}(3)$ by (see [3])

$$\tau(\chi) = \left(I_4 - \frac{1}{2}\hat{\chi}\right)^{-1}\left(I_4 + \frac{1}{2}\hat{\chi}\right) = \begin{bmatrix} \operatorname{cay}(\hat{\omega}) & \frac{4}{4+\|\omega\|^2}\left(I_3 + \frac{\hat{\omega}}{2} + \frac{\omega^T\omega}{4}\right)\gamma \\ 0 & 1 \end{bmatrix}$$

Likewise, τ is invertible for any $H = (R, r) \in SE(3)$ where R is not a rotation of angle $\pm\pi$, and the expression of its inverse is given by

$$\tau^{-1}(H) = -2\left(I_4 + H\right)^{-1}\left(I_4 - H\right) = \begin{bmatrix} \operatorname{cay}^{-1}(R) & 2(I_3 + R)^{-1}r \\ 0 & 0 \end{bmatrix}$$

Finally, we compute $\mathrm{d}\tau_\chi^{-1} : \mathbb{R}^6 \to \mathbb{R}^6$ the differential of τ^{-1} for any $\chi = (\omega, \gamma)$ (see [3])

$$\mathrm{d}\tau_\chi^{-1} = \begin{bmatrix} I_3 - \frac{1}{2}\hat{\omega} + \frac{1}{4}\omega\omega^T & 0 \\ -\frac{1}{2}\left(I_3 - \frac{1}{2}\hat{\omega}\right)\hat{\gamma} & I_3 - \frac{1}{2}\hat{\omega} \end{bmatrix}$$

Its dual $\mathrm{d}\tau_\chi^{-1\,*}$ on $\mathfrak{se}(3)^*$ is expressed as a matrix by the transpose of $\mathrm{d}\tau_\chi^{-1}$.

We approximate the reduced discrete Lagrangian on the triangle σ_i^j by

$$\ell_d^{i,j} := \ell_d(\psi_d(\sigma_i^j)) = hl\,\pounds\left(\chi_i^j, \epsilon_i^j\right)$$

Reformulating the variation of action (14.2.7) in our particular case, we obtain

$$\begin{aligned}
\delta\mathcal{A}_d(\psi_d) = &\sum_{i=1}^{N-1}\sum_{j=1}^{P-1}\Bigg((H_i^j)^{-1}D_H\ell_d^{i,j} + \frac{1}{h}\left(\mathrm{Ad}^*_{\tau(h\chi_{i-1}^j)}\,\mu_{i-1}^j - \mu_i^j\right)\\
&\qquad\qquad + \frac{1}{l}\left(\mathrm{Ad}^*_{\tau(l\epsilon_i^{j-1})}\,\nu_i^{j-1} - \nu_i^j\right)\Bigg)\cdot\zeta_i^j\\
&+\sum_{i=1}^{N-1}\left((H_i^0)^{-1}D_H\ell_d^{i,0} + \frac{1}{h}\left(\mathrm{Ad}^*_{\tau(h\chi_{i-1}^0)}\,\mu_{i-1}^0 - \mu_i^0\right) - \frac{1}{l}\nu_i^0\right)\cdot\zeta_i^0\\
&+\sum_{j=1}^{P-1}\left((H_0^j)^{-1}D_H\ell_d^{0,j} - \frac{1}{h}\mu_0^j + \frac{1}{l}\left(\mathrm{Ad}^*_{\tau(l\epsilon_0^{j-1})}\,\nu_0^{i-1} - \nu_0^j\right)\right)\cdot\zeta_0^j\\
&+\sum_{i=0}^{N-1}\frac{1}{l}\left(\mathrm{Ad}^*_{\tau(l\epsilon_i^{P-1})}\,\nu_i^{P-1}\right)\cdot\zeta_i^P + \sum_{j=0}^{P-1}\frac{1}{h}\left(\mathrm{Ad}^*_{\tau(h\chi_{N-1}^j)}\,\mu_{N-1}^j\right)\cdot\zeta_N^j\\
&+\left((H_0^0)^{-1}D_H\ell_d^{0,0} - \frac{1}{h}\mu_0^0 - \frac{1}{l}\nu_0^0\right)\cdot\zeta_0^0
\end{aligned}\tag{18.1.2}$$

where $D_H\ell_d^{i,j} = 0$ for any pair i,j. Applying the Hamilton principle on this Lagrangian yields the discrete Euler-Poincaré equations (14.2.11), that is $\forall i,j \in \{1,...,N-1\}\times\{1,...,P-1\}$,

$$\frac{1}{h}\left(\mathrm{Ad}^*_{\tau(h\chi_{i-1}^j)}\,\mu_{i-1}^j - \mu_i^j\right) + \frac{1}{l}\left(\mathrm{Ad}^*_{\tau(l\epsilon_i^{j-1})}\,\nu_i^{j-1} - \nu_i^j\right) = 0 \tag{18.1.3}$$

where μ_i^j and ν_i^j are the discrete momenta in $\mathfrak{se}(3)^*$ associated respectively to χ_i^j and ϵ_i^j and expressed by

$$\mu_i^j = \left(\mathrm{d}\tau_{h\chi_i^j}^{-1\,*}\right)D_\chi\ell_d^{i,j}, \qquad \nu_i^j = \left(\mathrm{d}\tau_{l\epsilon_i^j}^{-1\,*}\right)D_\epsilon\ell_d^{i,j}$$

Computing the partial derivatives of ℓ_d gives

$$D_\chi\ell_d^{i,j} = \mathbb{J}\chi_i^j \qquad D_\epsilon\ell_d^{i,j} = -\mathbb{C}(\epsilon_i^j - E_4)$$

where $\mathbb{J}$ and $\mathbb{C}$ are the inertial and Hook tensors defined respectively in (16.0.6) and (16.0.7). The momenta then become

$$\mu_i^j = \left(\mathrm{d}\tau_{h\chi_i^j}^{-1}\right)^T\mathbb{J}\chi_i^j, \qquad \nu_i^j = -\left(\mathrm{d}\tau_{l\epsilon_i^j}^{-1}\right)^T\mathbb{C}(\epsilon_i^j - E_4) \tag{18.1.4}$$

The expression of the equations on the boundary can be deduced from (18.1.2) by independence of the variations ζ_i^j

$$\forall i \in \{1,...,N-1\} \qquad \frac{1}{h}\left(\mathrm{Ad}^*_{\tau(h\chi_{i-1}^0)}\,\mu_{i-1}^0 - \mu_i^0\right) - \frac{1}{l}\nu_i^0 = 0 \tag{18.1.5a}$$

$$\forall j \in \{1,...,P-1\} \qquad -\frac{1}{h}\mu_0^j + \frac{1}{l}\left(\mathrm{Ad}^*_{\tau(l\epsilon_0^{j-1})}\,\nu_0^{j-1} - \nu_0^j\right) = 0 \tag{18.1.5b}$$

$$\forall i \in \{1, ..., N-1\} \qquad \frac{1}{l}\,\mathrm{Ad}^*_{\tau(l\epsilon_i^{P-1})}\,\nu_i^{P-1} = 0 \tag{18.1.5c}$$

$$\forall j \in \{1, ..., P-1\} \qquad \frac{1}{h}\,\mathrm{Ad}^*_{\tau(h\chi_{N-1}^{j})}\,\mu_{N-1}^{j} = 0 \tag{18.1.5d}$$

$$\frac{1}{l}\,\mathrm{Ad}^*_{\tau(l\epsilon_0^{P-1})}\,\nu_0^{P-1} = 0 \tag{18.1.5e}$$

$$\frac{1}{h}\,\mathrm{Ad}^*_{\tau(h\chi_{N-1}^{0})}\,\mu_{N-1}^{0} = 0 \tag{18.1.5f}$$

$$\frac{1}{h}\mu_0^0 + \frac{1}{l}\nu_0^0 = 0 \tag{18.1.5g}$$

We compute the discrete Cartan forms (14.2.9) and (14.2.10)

$$\Theta^1_{\ell_d}\left(H_i^j, \chi_i^j, \epsilon_i^j\right) = \left\langle -\frac{1}{h}\mu_i^j - \frac{1}{l}\nu_i^j, (H_i^j)^{-1}\mathrm{d}H_i^j \right\rangle \tag{18.1.6}$$

$$\Theta^2_{\ell_d}\left(H_i^j, \chi_i^j, \epsilon_i^j\right) = \left\langle \frac{1}{h}\,\mathrm{Ad}^*_{\tau(h\chi_i^j)}\,\mu_i^j, (H_{i+1}^j)^{-1}\mathrm{d}H_{i+1}^j \right\rangle \tag{18.1.7}$$

$$\Theta^3_{\ell_d}\left(H_i^j, \chi_i^j, \epsilon_i^j\right) = \left\langle \frac{1}{l}\,\mathrm{Ad}^*_{\tau(l\epsilon_i^j)}\,\nu_i^j, (H_i^{j+1})^{-1}\mathrm{d}H_i^{j+1} \right\rangle \tag{18.1.8}$$

and the associated discrete momenta (15.1.1) and (15.1.2)

$$J^1_{\ell_d}\left(H_i^j, \chi_i^j, \epsilon_i^j\right) = \mathrm{Ad}^*_{(H_i^j)^{-1}}\left(-\frac{1}{h}\mu_i^j - \frac{1}{l}\nu_i^j\right) \tag{18.1.9}$$

$$J^2_{\ell_d}\left(H_i^j, \chi_i^j, \epsilon_i^j\right) = \frac{1}{h}\,\mathrm{Ad}^*_{(H_i^j)^{-1}}\,\mu_i^j \tag{18.1.10}$$

$$J^3_{\ell_d}\left(H_i^j, \chi_i^j, \epsilon_i^j\right) = \frac{1}{l}\,\mathrm{Ad}^*_{(H_i^j)^{-1}}\,\nu_i^j \tag{18.1.11}$$

Since the fibre is a Lie group that acts on itself and leaves ℓ_d invariant, the discrete Noether theorem (6) applies.

18.2 Time-stepping resolution

The idea of the time resolution is to integrate the state of the beam at the next time step by integrating the current state along velocity at the current time step, that is

$$g_{i+1}^j = g_i^j \tau(h\chi_i^j) \tag{18.2.1}$$

For a time resolution, the state of the beam g_0^j and the velocity χ_0^j is given at time step $i = 0$ for all sections of the beam $j \in \{0, ..., P-1\}$. The state at time step $i = 1$ can then be computed by using (18.2.1). Since the state g_0^j is given, the variations on vertices $(0, j)$ vanish for all $j \in \{0, ..., P\}$. The other variations being free, conditions (18.1.5a) and (18.1.5c) have to be verified, which constitute the numerical method together with the discrete Euler-Poincaré equations (18.1.3). Here condition (18.1.5c) is ensured by taking $\epsilon_i^{P-1} = E_4\ \forall i \in \{1, ..., N-1\}$. An implementation of this numerical method is given by algorithm 1.

As did Demoures in [4], we can study the symplectic property of the time evolution of the solution. To do so, we need to reformulate the problem in a symplectic setting. This is done by defining a temporal discrete Lagrangian $\ell_d^t : G^P \times \mathfrak{g}^P \to \mathbb{R}$ obtained by summing ℓ_d along space at each time step

$$\ell_d^{t\ i} := \ell_d^t(H_i, \chi_i) = \sum_{j=0}^{P-1} \ell_d^{i,j}$$

Algorithm 1: Time resolution algorithm

Data: $\forall j \in \{0, P-1\}:\ g_0^j, \chi_0^j$
Initialisation:
for $j = 0$ **to** $P-1$ **do**
  $g_1^j = g_0^j \tau \left(h\chi_0^j \right)$
  $\mu_0^j = \left(\mathrm{d}\tau^{-1\ *}_{h\chi_0^j} \right) \mathbb{J} \chi_0^j$
end
Iteration:
for $i = 1$ **to** $N-2$ **do**
  for $j = 0$ **to** $P-1$ **do**
    if $j < P-1$ **then**
      $\epsilon_i^j = \frac{1}{l} \tau^{-1} \left((g_i^j)^{-1} g_i^{j+1} \right)$
    else
      $\epsilon_i^j = E_4$
    end
    $\nu_i^j = -\left(\mathrm{d}\tau^{-1\ *}_{l\epsilon_i^j} \right) \mathbb{C}(\epsilon_i^j - E_4)$
    if $j = 0$ **then**
      $\mu_i^0 = \mathrm{Ad}^*_{\tau(h\chi_{i-1}^0)}\, \mu_{i-1}^0 - \frac{h}{l} \nu_i^0$
    else if $j = P-1$ **then**
      $\mu_i^{P-1} = \mathrm{Ad}^*_{\tau(h\chi_{i-1}^{P-1})}\, \mu_{i-1}^{P-1} + \frac{h}{l}\, \mathrm{Ad}^*_{\tau(l\epsilon_i^{P-2})}\, \nu_i^{P-2}$
    else
      $\mu_i^j = \mathrm{Ad}^*_{\tau(h\chi_{i-1}^j)}\, \mu_{i-1}^j + \frac{h}{l} \left(\mathrm{Ad}^*_{\tau(l\epsilon_i^{j-1})}\, \nu_i^{j-1} - \nu_i^j \right)$
    end
    find χ_i^j **such that** $\chi_i^j = \mathbb{J}^{-1} \left(\mathrm{d}\tau^{-1}_{h\chi_i^j} \right) \mu_i^j$
    $g_{i+1}^j = g_i^j \tau \left(h\chi_i^j \right)$
  end
end

where H_i is the vector $(H_i^0, ..., H_i^{P-1})$ and $\chi_i = (\chi_i^0, ..., \chi_i^{P-1})$. The associated action map $\mathcal{A}_d$ is given by

$$\mathcal{A}_d(H) = \sum_{i=0}^{N-1} \ell_d^t(H_i, \chi_i)$$

The variation of action is expressed by

$$\begin{aligned}
\delta\mathcal{A}_d(\psi_d) &= \sum_{i=0}^{N-1} D_H \ell_d^t(H_i, \chi_i) \cdot \delta H_i + D_\chi \ell_d^t(H_i, \chi_i) \cdot \delta\chi_i \\
&= \sum_{i=1}^{N-1} \frac{1}{h} \left(\mathrm{Ad}^*_{\tau(h\chi_{i-1})}\, \mu_{i-1} - \mu_i \right) \cdot \zeta_i + \frac{1}{h} \mu_0 \cdot \zeta_0 + \frac{1}{h}\, \mathrm{Ad}^*_{\tau(h\chi_{N-1})}\, \mu_{N-1} \cdot \zeta_N
\end{aligned}$$

Applying the Hamilton principle yields the discrete Euler-Poincaré equations $\forall i \in \{1, ..., N-1\}$

$$\mathrm{Ad}^*_{\tau(h\chi_{i-1})}\,\mu_{i-1} = \mu_i \tag{18.2.2}$$

which are equivalent to equations (18.1.3), (18.1.5a) and (18.1.5c). We also deduce from the variation of action the boundary conditions

$$\mu_0 = 0 \tag{18.2.3}$$
$$\mathrm{Ad}^*_{\tau(h\chi_{N-1})}\,\mu_{N-1} = 0 \tag{18.2.4}$$

which are respectively equivalent to (18.1.5b), (18.1.5e), (18.1.5g) and (18.1.5d), (18.1.5f).

We define the Poincaré-Cartan forms $\Theta^+_{\ell^t_d} := \Theta^2_{\ell^t_d}$ and $\Theta^-_{\ell^t_d} := -\Theta^1_{\ell^t_d}$ on $G^P \times \mathfrak{g}^P$ by applying (14.2.9) and (14.2.10)

$$\Theta^+_{\ell^t_d}(H_i, \chi_i) = \left\langle \frac{1}{h}\,\mathrm{Ad}^*_{\tau(h\chi_i)}\,\mu_i, H^{-1}_{i+1}\mathrm{d}H_{i+1} \right\rangle = \sum_{j=0}^{P-1} \Theta^2_{\ell_d}\left(H^j_i, \chi^j_i, \epsilon^j_i\right)$$

$$\Theta^-_{\ell^t_d}(H_i, \chi_i) = -\left\langle -\frac{1}{h}\mu_i, H^{-1}_i\mathrm{d}H_i \right\rangle = \sum_{j=0}^{P-1} -\Theta^1_{\ell_d}\left(H^j_i, \chi^j_i, \epsilon^j_i\right) - \Theta^3_{\ell_d}\left(H^j_i, \chi^j_i, \epsilon^j_i\right)$$

We also define the momentum map $J^+_{\ell^t_d} := J^2_{\ell^t_d}$ and $J^-_{\ell^t_d} := -J^1_{\ell^t_d}$ on $G^P \times \mathfrak{g}^P \to \mathfrak{g}^*$ by applying (15.1.1) and (15.1.2)

$$J^+_{\ell^t_d}(H_i, \chi_i) = \sum_{j=0}^{P-1} J^2_{\ell_d}\left(H^j_i, \chi^j_i, \epsilon^j_i\right)$$

$$J^-_{\ell^t_d}(H_i, \chi_i) = \sum_{j=0}^{P-1} -J^1_{\ell_d}\left(H^j_i, \chi^j_i, \epsilon^j_i\right) - J^3_{\ell_d}\left(H^j_i, \chi^j_i, \epsilon^j_i\right)$$

Since the discrete local Noether theorem (6) holds for the Lagrangian ℓ^t_d, we get $J^+_{\ell^t_d} = J^-_{\ell^t_d} =: J_{\ell^t_d}$ and we compute

$$J_{\ell^t_d}(H_i, \chi_i) = \sum_{j=0}^{P-1} \frac{1}{l}\,\mathrm{Ad}^*_{(H^j_i)^{-1}}\,\mu^j_i \tag{18.2.5}$$

Figure 11 shows the computation of the coefficients of $J_{\ell^t_d}$ against time. One can see that those are preserved up to a round-off error (the precision used for the coefficients of the Lie group and algebra elements is 10^{-16}; see appendix C for a more extensive discussion on the mathematical objects representation in the implementation).

In the case where a boundary condition is given at time $i = 0$, H_0 is fixed and therefore $\zeta_0 = 0$. The discrete Euler-Poincaré equations (18.2.2) holds since ζ_i is arbitrary for all $i > 0$, hence the discrete flow map $F_{\ell^t_d} : G^P \times \mathfrak{g}^P \to G^P \times \mathfrak{g}^P$ corresponding to the discrete Euler-Poincaré equations leaves the symplectic form $\Omega_{\ell^t_d} = \mathrm{d}\Theta^+_{\ell^t_d}$ invariant

$$F^*_{\ell^t_d}\Omega_{\ell^t_d} = \Omega_{\ell^t_d} \tag{18.2.6}$$

We recall from the example in section 11 that the symplecticity implies that the energy of the system is approximately conserved; hence, integrating the discrete solution of the beam with time boundary conditions should leave the energy approximately invariant.

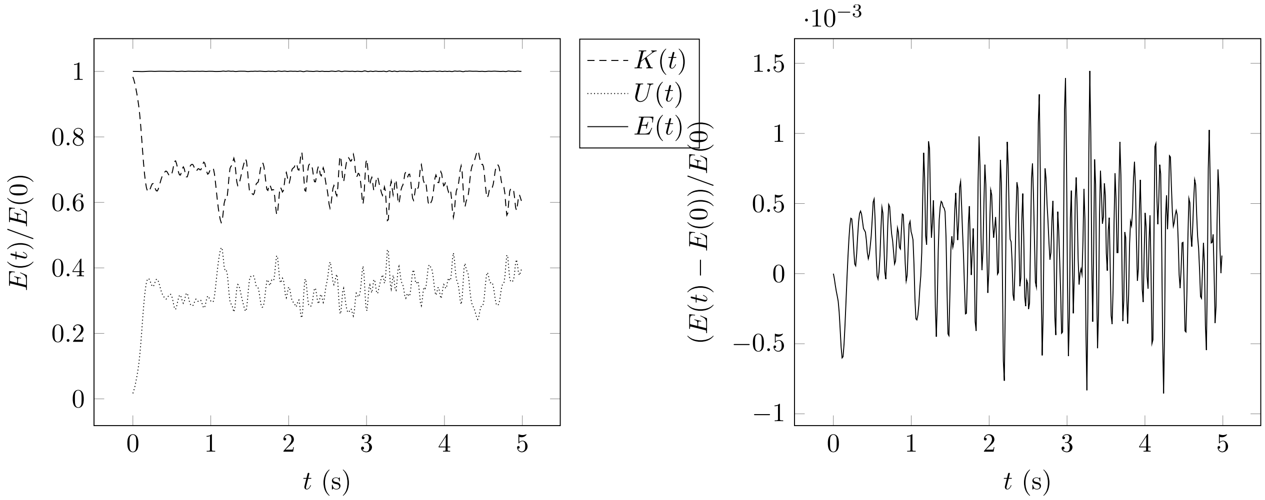
We define the discrete energy $E_{\ell^t_d} : G^P \times \mathfrak{g}^P \to \mathbb{R}$ as

$$E_{\ell^t_d}(H_i, \chi_i) = \sum_{j=0}^{P-1} K(\chi^j_i) + U(\epsilon^j_i) = \sum_{j=0}^{P-1} \frac{1}{2}\,{\chi^j_i}^T\,\mathbb{J}\chi^j_i + \frac{1}{2}(\epsilon^j_i - E_4)^T\mathbb{C}(\epsilon^j_i - E_4) \tag{18.2.7}$$

Figure 10 shows that the energy computed for the numerical example is indeed approximately conserved as expected, with a relative error that seem to be of order 10^{-3}. Figure 12 shows the same result over a 30 minutes computation, which represent 1.8×10^6 time steps of duration $0.001s$.

Numerical results The numerical resolution of the Reissner beam has been computed for the following parameters, that roughly correspond to the physical properties of a $1m$ long and $10cm$ diameter rubber beam :

$$h = 0.001s;\ l = 0.1m;\ L = 1m$$
$$E = 1.0 \times 10^4 Pa;\ \nu = 0.3;\ \rho = 1.5 \times 10^3 kg/m^3;\ a = 0.05m$$



(a) Discrete kinetic, potential and total energies.

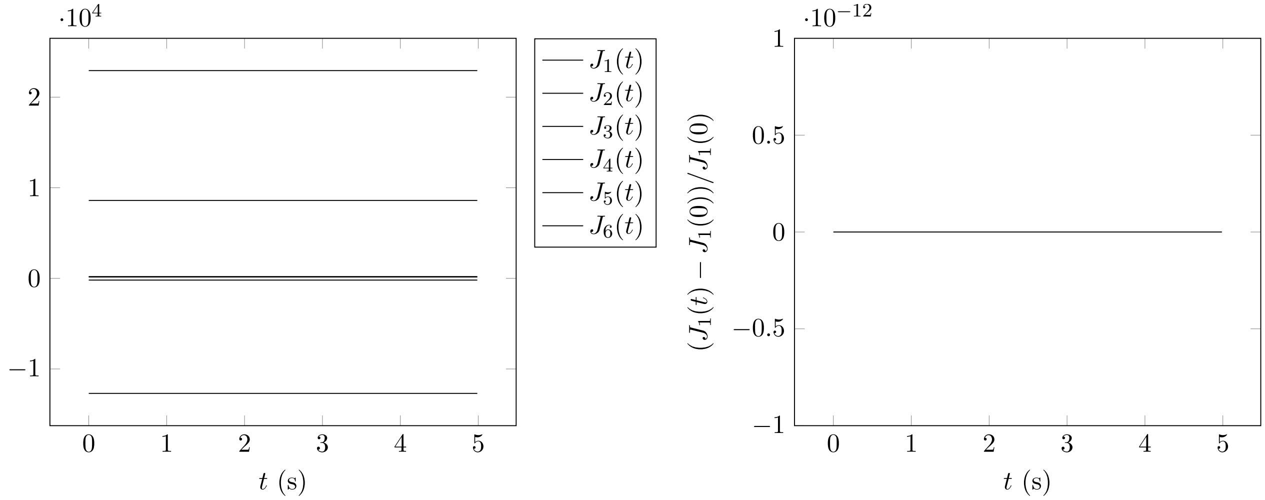
(b) Relative error of discrete total energy.

Figure 10 – Approximate conservation of energy over $5s$.

Remark At each iteration of the main loop of algorithm (1), one need to find χ such that

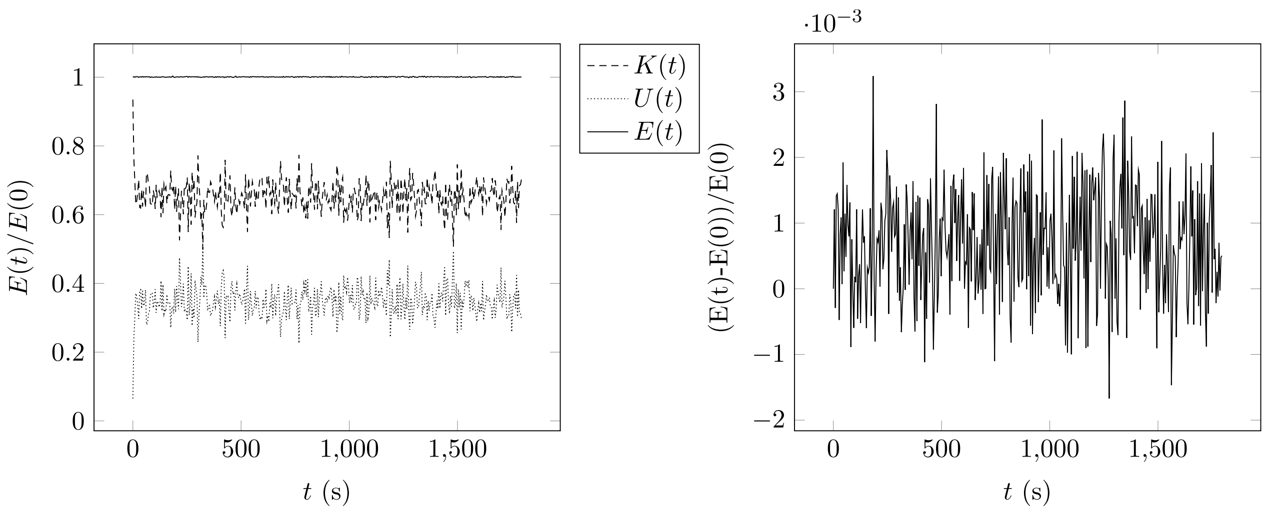
$$\mu = \left(\mathrm{d}\tau_{h\chi}^{-1\,*} \right) \mathbb{J}\chi \tag{18.2.8}$$

for a given μ. The implementation of this resolution has been done by solving the non linear 6 by 6 system of equations $F(\chi) = 0$ with F the difference between the right and left members of equation (18.2.8) with external libraries. This non linear system has not be proved by us to have solutions in the general case, and therefore the solving is done by supposing the existence and uniqueness of such a solution – at least in the neighbourhood of the heuristic initial value χ_0. For this reason, the convergence of the solver strongly relies on the parameters of the problem. In particular, the relation between the density ρ and the Young modulus E is a predominant factor of the convergence of the solver. In the case where ρ and E are of similar magnitude, the algorithm does well and solves the system with an arbitrary precision, but for E bigger than ρ by several orders of magnitude, the solver is incapable to complete at some nodes, even with very rough precision for the zero solving (10^{-3}). For this reason, the examples presented are limited to the case where the young modulus is small (between 10^3 and 10^5), which narrows the scope of the possible applications. More detail is given on the implementation in appendix C.



(a) Discrete momenta $J^i_{\ell^t_d}$ (b) Relative error of discrete momentum $J^1_{\ell^t_d}$

Figure 11 – Exact conservation of discrete momenta over $5s$.



(a) Discrete kinetic, potential and total energies. (b) Relative error of discrete total energy.

Figure 12 – Approximate conservation of energy for a 30min computation.

18.3 Space-stepping resolution

The space resolution is done by integrating the state of the beam at the next space step along deformation at the current time step, that is

$$g_i^{j+1} = g_i^j \tau(l\epsilon_i^j) \tag{18.3.1}$$

The state of a section of the beam g_i^0 and its velocity ϵ_i^0 are given at all time steps $i \in \{0, ..., N-1\}$. The state at space step $j = 1$ can then be computed by using (18.3.1). The numerical method is obtain by verifying conditions (18.1.5b) and (18.1.5d) together with the discrete Euler-Poincaré equations (18.1.3). Condition (18.1.5c) is ensured by taking $\chi^j_{N-1} = 0 \ \forall j \in \{1, ..., P-1\}$. The obtained algorithm is the same as algorithm 1 where the role played by χ, ϵ and μ, ν are inverted.

Since the implementation is very similar, it is not explicitly given here, the reader should refer to the time algorithm and apply the appropriate modifications.

In the case where a boundary condition is given at section $j = 0$, ζ_i^0 is equal to zero for all i, hence we are not ensured that equation (18.1.5b) is true. The discrete Euler-Poincaré equations (18.2.2) does not hold any more, and the associated discrete flow is not ensured to be symplectic with respect to $\Omega_{\ell_d^t}$; therefore, the energy may not be conserved in the case of a space boundary condition. In the same manner, the discrete Noether theorem does not hold in time since the discrete Euler-Poincaré equations (18.2.2) are not verified, hence the momentum map $J_{\ell_d^t}$ is not conserved over time any more.

The same treatment could be applied to a time integrated Lagrangian that would lead to similar results on space symplecticity and momentum conservation, but has not been done since the ideas are the same and the result of the conservation of a symplectic form along space and the approximative conservation of a "spatial" energy do not have any physically meaningful interpretation.

19 Application to sound synthesis

In this section we shall apply the results of the discrete modelling of the Reissner beam to sound synthesis. The modelling of a string by a Reissner beam is valid in the case of large displacements as long as the system stays in the elastic domain, which is the case in the setting of musical instruments for example.

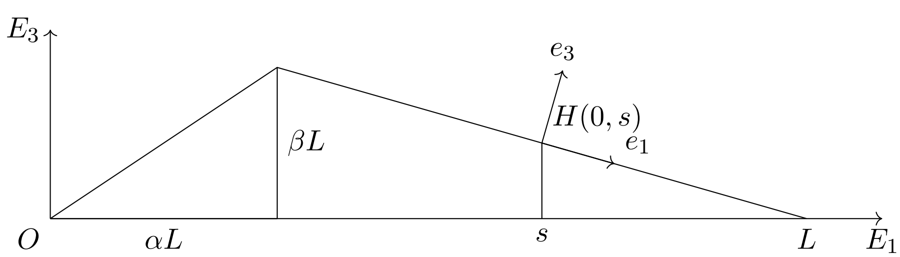
We shall point out that the results exposed here are not usable as is for sound synthesis since, as pointed out by remark 18.2, the physical properties of the material, in particular ρ and E, do not correspond to those used for a typical string (guitar, piano, violin, etc) which would be made of nylon or steel; this is due to a problem of convergence of the solver. However, the principles presented here are the same regardless the physical properties of the material.

19.1 Case A : plucked string

In this application, we focus on a time stepping algorithm, with homogeneous Dirichlet condition on space boundary, and Dirichlet and Neumann conditions on time boundary. In other words, we suppose that the string is fixed at the endpoints, and that the position and speed of the string is prescribed at $t = 0$. We choose the following conditions

$$\begin{aligned} H_i^0 &= (\mathrm{Id}_3, (0,0,0))\ \forall i \in \{1, ..., N\} \\ H_i^P &= (\mathrm{Id}_3, (L,0,0))\ \forall i \in \{1, ..., N\} \\ \chi_0^j &= 0_{\mathfrak{se}(3)}\ \forall j \in \{1, ..., P\} \end{aligned}$$

and we prescribe the position of the beam H_0 at $t = 0$ by



where $\alpha \in]0, 1[$ and $\beta \geq 0$. There is no translation along E_2 component, nor any rotation on another axis than E_2, hence the beam should stay in the plane spanned by E_1 and E_3 at every time t. Expressed explicitly, we have for H_0 :

$$\hat{H}(0,s) = \begin{bmatrix} \cos(\theta_a) & 0 & \sin(\theta_a) & s \\ 0 & 1 & 0 & 0 \\ -\sin(\theta_a) & 0 & \cos(\theta_a) & \beta\frac{s}{\alpha} \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ if } s \in]0, \alpha L[$$

$$\hat{H}(0,s) = \begin{bmatrix} \cos(\theta_b) & 0 & \sin(\theta_b) & s \\ 0 & 1 & 0 & 0 \\ -\sin(\theta_b) & 0 & \cos(\theta_b) & \beta\left(L - \frac{s-\alpha L}{1-\alpha}\right) \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ if } s \in]\alpha L, L[$$

where $\theta_a = -\alpha/\sqrt{\alpha^2+\beta^2}$ and $\theta_b = (1-\alpha)/\sqrt{(1-\alpha)^2+\beta^2}$. This type of configuration corresponds to an excitation of the string obtained by pulling it at a unique point and releasing it at $t = 0$ without any initial speed.

To perform a time stepping integration, the algorithm (1) has to be slightly modified in order to take into account the homogeneous Dirichlet in space boundary conditions. Since H_i^0 and H_i^P are prescribed and constant for all i, variations are null on the corresponding nodes, hence we do not need to verify equations (18.1.5a) nor (18.1.5c), and since the position is prescribed at $t = 0$ we do not need to verify (18.1.5b), (18.1.5e) and (18.1.5g) either. Finally, we are just left with the discrete Euler-Poincaré equations (18.1.3).

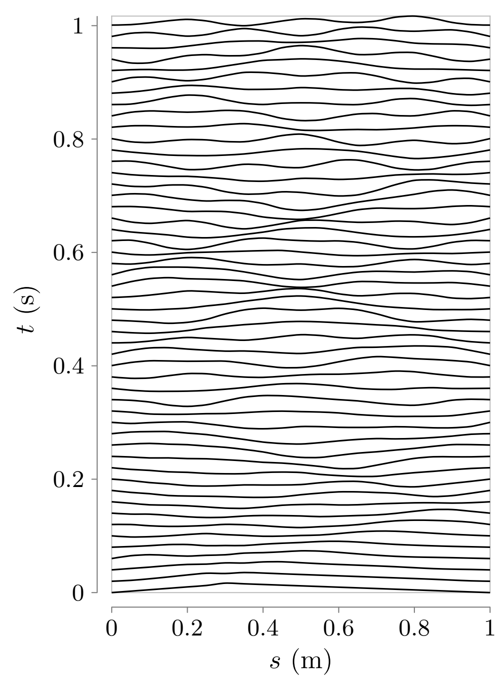


Figure 13 – Time evolution of the string over a $1s$ time period.

The result of the iteration of the algorithm with the previous boundary conditions is given for the following parameters :

$$h = 0.001s;\ l = 0.05m;\ L = 1m$$
$$E = 5.0 \times 10^4 Pa;\ \nu = 0.35;\ \rho = 1.0 \times 10^3 kg.m^{-3};\ a = 0.01m;$$
$$\alpha = 1/3;\ \beta = 0.01;$$

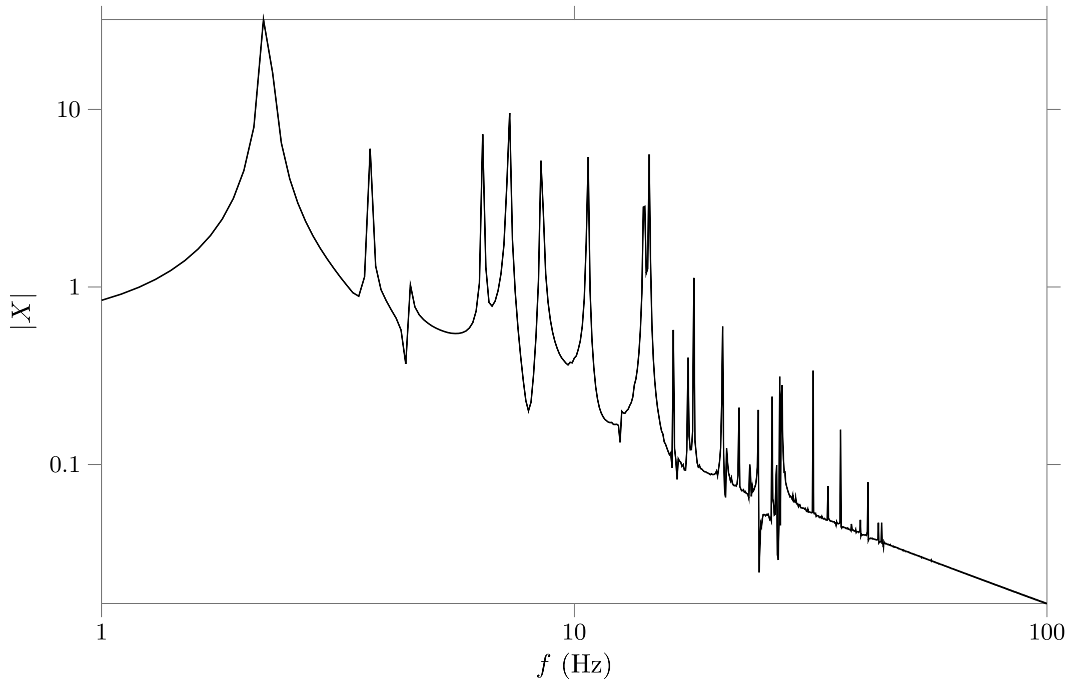


Figure 14 – Amplitude of the spectral analysis of the vibration of the string.

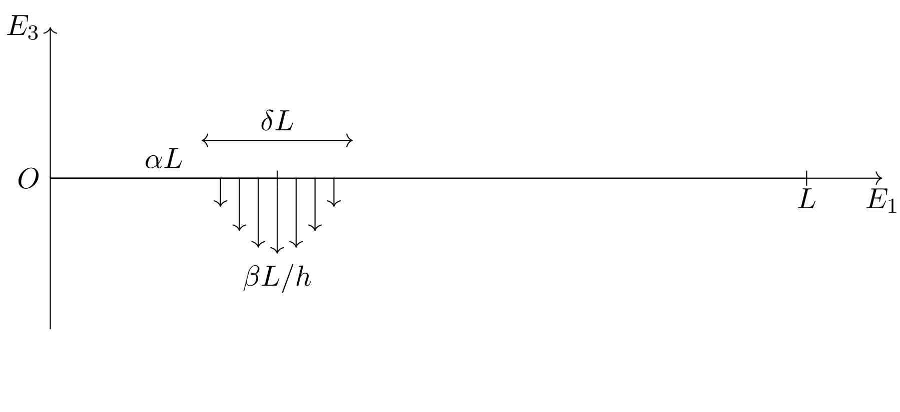
The time evolution of the string is represented in figure 13. One can clearly see the right and left propagating waves that reflect on the endpoints. The spectral analysis of 30s of signal given in figure 14 shows that the signal has well-defined partials, but is clearly not harmonic. This is consistent with the fact that the system is intrinsically non linear.

19.2 Case B : hammered string

We take here the same boundary conditions, but with a different initial value. We suppose the string is at the equilibrum at $t < 0$, and we impose the initial velocity; we obtain

$$H_i^0 = (\mathrm{Id}_3, (0,0,0)) \ \forall i \in \{1, ..., N\}$$
$$H_i^P = (\mathrm{Id}_3, (L,0,0)) \ \forall i \in \{1, ..., N\}$$
$$H_0^j = (\mathrm{Id}_3, (0,0,0)) \ \forall j \in \{1, ..., P\}$$

where the velocity of the beam χ_0 at $t = 0$ is given by



where $\alpha \in]0,1[$, $\beta \geq 0$ and $\delta \in]0,\alpha[$. There is again no translation along E_2 component, nor any rotation, hence the beam should stay in the plane spanned by E_1 and E_3 at every time t. The explicit expression for χ_0 is

$$\chi(0,s) = 0_{\mathfrak{se}(3)} \text{ if } s \in \left]0, (\alpha - \frac{\delta}{2})L\right[\cup \left](\alpha + \frac{\delta}{2})L, L\right[$$
$$\chi(0,s) = \left(0,0,0,0,0,-\frac{\beta L}{h}\cos\left(\pi\frac{s/L-\alpha}{\delta}\right)\right) \text{ if } s \in [(\alpha - \frac{\delta}{2})L, (\alpha + \frac{\delta}{2})L]$$

This type of configuration corresponds to an excitation of the string obtained by hammering it, as for a piano for example.

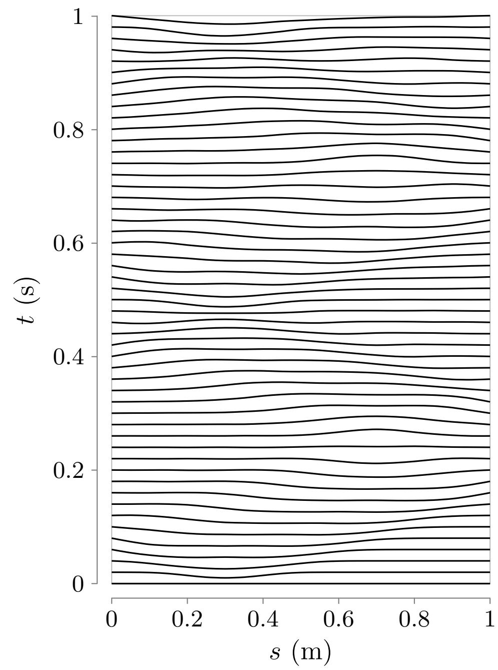


Figure 15 – Time evolution of the string over a $1s$ time period.

The result of the iteration of the algorithm is given for the same physical properties. Here we chose

$$\alpha = 1/3;\ \beta = 3.5 \times 10^{-4};\ \delta = 0.3;$$

We observe on figure 15 the left and right propagating waves again, and figure 16 shows that the first and more prominent frequency of the signal is the same as for figure 14, which is consistent with the fact that the physical properties of the system are unchanged between the two examples; however, the partials are different.

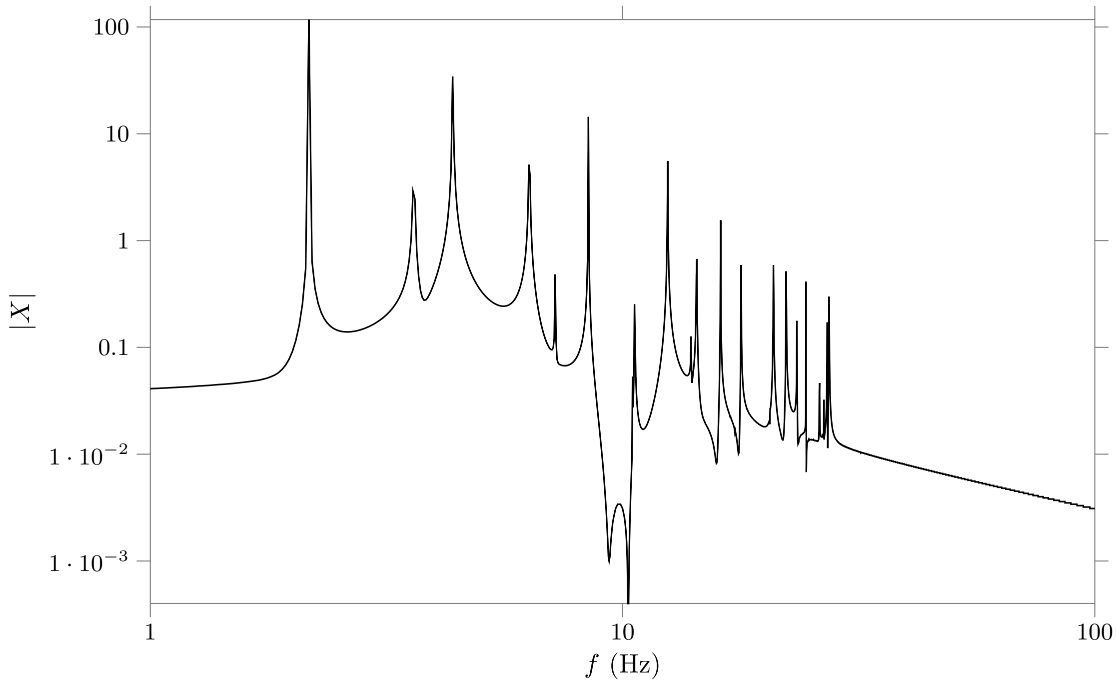


Figure 16 – Amplitude of the spectral analysis of the vibration of the string.

Part VIII
Conclusion

20 Summary

After a state of the art on the Lagrangian multisymplectic theory, we proposed a synthetic and general method to design structure preserving multisymplectic integrators. We then investigated the case where the configuration space of the physical system is a Lie group, and introduced a way to build Lie group integrators in a particular choice of base space discretization. Finally, we applied this formalism in the 2-dimensional case of the Reissner beam, and proposed some applications in sound synthesis by modelling a vibrating string by a Reissner beam.

21 Perspectives

Starting from this report, several perspectives can be outlined :

– Because of their long time behaviour, the methods exposed in this report are of particular interest for HPC. The challenges faced by their implementation for large scale computing are to obtain a massively parallel implementation of the algorithms, to integrate them together with the Port Hamiltonian System in order to divide the physical system in parallel I/O subsystems, and to make efficient Lie groups computations.

– The methods we studied are only defined for conservative, force and torque free systems. An important point for concrete applications would be to develop an algorithm that takes forces into account. This has already been done in particular cases, for example on a 2-D base space case with a potential by Demoures [4].

– On a smaller scale, our implementation of the Reissner beam shows promising results, but is currently unable to perform simulations interesting for musical sound synthesis, and its computation time makes it unable to run in real time. This problem could be overcome by implementing a more efficient solving of the non-linear system. Interesting sound synthesis applications would also need the modelling of interaction with other systems and dissipation effects, which require to modify the current model that does not take them into account.

A $SE(3)$ memento

In the matrix representation, elements $\hat{H}$ of the group $(SE(3), \times)$ are represented by

$$\hat{H} = \begin{bmatrix} R & r \\ 0 & 1 \end{bmatrix} \in \mathcal{M}_{4\times 4}(\mathbb{R})$$

where $R \in \mathcal{M}_{3\times 3}(\mathbb{R})$ is a rotation matrix (that can be considered as an element of the three dimensional rotation group $SO(3)$) and $r \in \mathbb{R}^3$ is a translation vector. Since the representation of R in a vector form is not unique and depends on the choice of a particular set of Euler angles, we never explicitly use this representation, and stick with the matrix representation; the same remark holds for $\hat{H}$.

The product of two elements of the group $\hat{H}_1$ and $\hat{H}_2$ is given by

$$\hat{H}_1 \times \hat{H}_2 = \begin{bmatrix} R_1 R_2 & R_1 r_2 + r_1 \\ 0 & 1 \end{bmatrix}$$

and the invert of an element $\hat{H}$ is given by

$$\hat{H}^{-1} = \begin{bmatrix} R^T & -R^T r \\ 0 & 1 \end{bmatrix}$$

In the matrix representation, elements ξ of the Lie algebra $(\mathfrak{se}(3), +)$ are represented by

$$\hat{\chi} = \begin{bmatrix} \hat{\omega} & \gamma \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_{4\times 4}(\mathbb{R})$$

where $\hat{\omega} \in \mathcal{M}_{3\times 3}(\mathbb{R})$ is an element of $\mathfrak{so}(3)$ and $\gamma \in \mathbb{R}^3$. Here $\hat{\omega}$ is a skew symmetric matrix, hence can be written in the form

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} =: \omega \cdot J$$

where we define $J := (J_1, J_2, J_3)$ by

$$J_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad J_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \qquad J_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

Here ω is the vector representation of the element of $\mathfrak{so}(3)$, and its matrix representation is obtained through the hat map operation $\hat{\cdot} : \mathbb{R}^3 \to \mathcal{M}_{3\times 3}(\mathbb{R})$ such that $\hat{\omega} = \omega \cdot J$. We define the vector representation of $\mathfrak{se}(3)$ elements by

$$\chi = (\omega, \gamma) \in \mathbb{R}^6.$$

The composition of two elements χ_1 and χ_2 of the algebra is given by

$$\chi_1 + \chi_2 = (\omega_1 + \omega_2, \gamma_1 + \gamma_2)$$

and the inverse of an element χ by

$$\chi^{-1} = -\chi = (-\omega, -\gamma)$$

We also give the expressions in vector form of the operators

$$\mathrm{Ad} : SE(3) \times \mathfrak{se}(3) \to \mathfrak{se}(3)$$

$$\mathrm{Ad}^* : SE(3) \times \mathfrak{se}(3)^* \to \mathfrak{se}(3)^*$$

where Ad^* is defined by the pairing $\langle \cdot, \cdot \rangle : \mathfrak{se}(3)^* \times \mathfrak{se}(3) \to \mathbb{R}$

$$\langle \mathrm{Ad}^* \cdot, \cdot \rangle = \langle \cdot, \mathrm{Ad} \cdot \rangle$$

We obtain for $H = (R, r)$, $\chi = (\omega, \gamma)$ and $\pi = (\mu, \beta)$:

$$\mathrm{Ad}_H \chi = (R\omega, -R\omega \times r + R\gamma)$$
$$\mathrm{Ad}^*_{H^{-1}} \pi = (R\mu + r \times R\beta, R\beta)$$

B Proofs and computations

B.1 Proof of proposition (2)

Let $Z \in \chi(E)$ be an arbitrary vector field with $Z = \alpha^\mu \vec{\partial}_\mu + \beta^A \vec{\partial}_A$ and let $\bar{\mathrm{P}} = (x^\mu, y^A, v^A_\mu)$ be a point of J^1E denoted in a local coordinate system, for each $\bar{\mathrm{P}}$ is associated the couple $(\mathrm{P}, (X_\mu|_\mathrm{P}))$ of P point of E and $(X_\mu|_\mathrm{P})$ a basis of $T_\mathrm{P}E$ given by $\mathrm{P} = \pi^1(\bar{\mathrm{P}})$ and $\forall \mu \in \{1, ..., n+1\}$, $\theta_{\bar{\mathrm{P}}}(X_\mu|_\mathrm{P}) = 0$ and $\mathrm{d}x_\nu(X_\mu) = \delta^\nu_\mu$. With X_μ defined in that manner, it can be written $X_\mu = \vec{\partial}_\mu + v^A_\mu \vec{\partial}_A$ (remark: $\forall \mu \in \{1, ..., n+1\}$, $X_\mu \in \chi(E)$ is a vector field, defined at point $\mathrm{P} \in E$ by the vector $X_\mu|_\mathrm{P}$).

Consider a one-parameter transformation group τ^Z_ε along Z parametrized by ε, let $\mathrm{P}_\varepsilon = \tau^Z_\varepsilon(\mathrm{P})$ be a point of E and $X^\varepsilon_\mu|_{\mathrm{P}\varepsilon} = T_\mathrm{P}\tau^Z_\varepsilon(X_\mu)$ a vector of $T_{\mathrm{P}_\varepsilon}E$, and construct $\bar{\mathrm{P}}_\varepsilon$ such that $\pi^1(\bar{\mathrm{P}}_\varepsilon) = \mathrm{P}_\varepsilon$ and $\forall \mu \in \{1, ..., n+1\}$, $\theta_{\bar{\mathrm{P}}_\varepsilon}(X^\varepsilon_\mu|_{\mathrm{P}_\varepsilon}) = 0$; then j^1Z is defined at point $\bar{\mathrm{P}}$ by

$$j^1Z(\bar{\mathrm{P}}) = \lim_{\varepsilon \to 0} \frac{\bar{\mathrm{P}}_\varepsilon - \bar{\mathrm{P}}}{\varepsilon} = \alpha^\mu \vec{\partial}_\mu + \beta^A \vec{\partial}_A + \lim_{\varepsilon \to 0} \frac{(v_\varepsilon)^A_\mu - v^A_\mu}{\varepsilon}$$

with $\mathrm{d}y^A(X^\varepsilon_\mu|_{\mathrm{P}_\varepsilon}) = (v_\varepsilon)^A_\nu \mathrm{d}x^\nu(X^\varepsilon_\mu|_{\mathrm{P}_\varepsilon})$.

We introduce the Lie derivative of X_μ at point P with respect to Z

$$L_Z X_\mu(\mathrm{P}) = [Z, X_\mu]_\mathrm{P} = \lim_{\varepsilon \to 0} \frac{T_{\mathrm{P}_\varepsilon}\tau^Z_{-\varepsilon}(X_\mu) - X_\mu|_\mathrm{P}}{\varepsilon}$$

Taking the Lie derivative of Z at point P_ε with respect to X_μ, $X^\varepsilon_\mu|_{\mathrm{P}_\varepsilon}$ can be expressed as

$$X^\varepsilon_\mu|_{\mathrm{P}\varepsilon} = X_\mu|_{\mathrm{P}\varepsilon} + \varepsilon \, [X_\mu, Z]_{\mathrm{P}_\varepsilon} + \mathcal{O}(\varepsilon^2)$$

which yields, X^ε_μ being normalised along $\vec{\partial}_\nu$,

$$(v_\varepsilon)^A_\mu - v^A_\mu = \varepsilon \theta^A_{\bar{\mathrm{P}}_\varepsilon} \left([X_\mu, Z]_{\mathrm{P}_\varepsilon} \right) + \mathcal{O}(\varepsilon^2)$$

From this comes

$$\mathrm{d}v^A_\mu(j^1Z) = \lim_{\varepsilon \to 0} \frac{(v_\varepsilon)^A_\mu - v^A_\mu}{\varepsilon} = \theta^A_{\bar{\mathrm{P}}_\varepsilon} \left([X_\mu, Z]_{\mathrm{P}_\varepsilon} \right)$$

We compute the Lie bracket for any function f

$$\begin{aligned}
[X_\mu, Z]f &= X_\mu \left(\alpha_\nu \vec{\partial}_\nu + \beta^A \vec{\partial}_A \right) f - Z \left(\vec{\partial}_\mu + v^B_\mu \vec{\partial}_B \right) f \\
&= \left(\vec{\partial}_\mu + v^B_\mu \vec{\partial}_B \right) \left(\alpha_\nu \frac{\partial f}{\partial x^\nu} + \beta^A \frac{\partial f}{\partial y^A} \right) - \left(\alpha_\nu \vec{\partial}_\nu + \beta^A \vec{\partial}_A \right) \left(\frac{\partial f}{\partial x^\mu} + v^B_\mu \frac{\partial f}{\partial y^B} \right) \\
&= \left(\frac{\partial \alpha_\nu}{\partial x^\mu} + v^B_\mu \frac{\partial \alpha_\nu}{\partial y^B} \right) \frac{\partial f}{\partial x^\nu} + \left(\frac{\partial \beta^A}{\partial x^\mu} + v^A_\mu \frac{\partial \beta^A}{\partial y^B} \right) \frac{\partial f}{\partial y^A}
\end{aligned}$$

This yields

$$\begin{aligned}
\gamma^A_\mu = \theta^A_{\bar{\mathrm{P}}_\varepsilon} \left([X_\mu, Z]_{\mathrm{P}_\varepsilon} \right) &= \left(\frac{\partial \beta^A}{\partial x^\mu} + v^A_\mu \frac{\partial \beta^A}{\partial y^B} \right) - v^A_\nu \left(\frac{\partial \alpha_\nu}{\partial x^\mu} + v^B_\mu \frac{\partial \alpha_\nu}{\partial y^B} \right) \\
&= \frac{\partial}{\partial x^\mu} \left(\beta^A - v^A_\nu \alpha_\nu \right) + v^B_\mu \frac{\partial}{\partial y^B} \left(\beta^A - v^A_\nu \alpha_\nu \right) = \frac{\partial \zeta^A}{\partial x^\mu} + v^B_\mu \frac{\partial \zeta^A}{\partial y^B}
\end{aligned}$$

where $\zeta^A = \beta^A - v^A_\nu \alpha^A$, which proves the proposition.

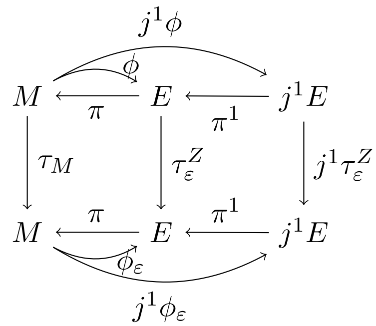
B.2 Computation of (6.3.2)

In order to find the stationary sections of the action map $\mathcal{A}$, one need to look for the sections ϕ that realise

$$\delta\mathcal{A}(\phi) = \lim_{\varepsilon\to 0} \frac{\mathcal{A}(\phi_\varepsilon) - \mathcal{A}(\phi)}{\varepsilon} = 0$$

where ϕ_ε is a variation of ϕ given by a one-parameter transformation group associated to an arbitrary vector field $Z \in \chi(E)$.

Let $\tau_M : M \to M$ be the diffeomorphism induced by τ_ε^Z so that $\tau_\varepsilon^Z \circ \phi = \phi_\varepsilon \circ \tau_M$; the following diagram commutes :



Then for any $\phi : M \supset \mathcal{U} \to E$, the variation of action $\delta\mathcal{A}$ evaluated on ϕ varying along Z is given by

$$\begin{aligned}
\delta\mathcal{A} &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\int_{\tau_M(\mathcal{U})} (j^1\phi_\varepsilon)^*\mathcal{L} - \int_{\mathcal{U}} (j^1\phi)^*\mathcal{L}\right) \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\int_{\tau_M(\mathcal{U})} (j^1\tau_\varepsilon^Z \circ j^1\phi \circ \tau_M^{-1})^*\mathcal{L} - \int_{\mathcal{U}} (j^1\phi)^*\mathcal{L}\right) \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\int_{\tau_M(\mathcal{U})} (\tau_M^{-1})^*(j^1\phi)^*(j^1\tau_\varepsilon^Z)^*\mathcal{L} - \int_{\mathcal{U}} (j^1\phi)^*\mathcal{L}\right) \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\int_{\mathcal{U}} (j^1\phi)^*(j^1\tau_\varepsilon^Z)^*\mathcal{L} - \int_{\mathcal{U}} (j^1\phi)^*\mathcal{L}\right) \\
&= \int_{\mathcal{U}} (j^1\phi)^*\left(\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[(j^1\tau_\varepsilon^Z)^*\mathcal{L} - \mathcal{L}\right]\right) = \int_{\mathcal{U}} (j^1\phi)^* L_{j^1Z}\mathcal{L} \\
&= \int_{\mathcal{U}} (j^1\phi)^* d(j^1Z \lrcorner \mathcal{L}) + \int_{\mathcal{U}} (j^1\phi)^* j^1Z \lrcorner d\mathcal{L} \\
&= \int_{\partial\mathcal{U}} (j^1\phi)^*(j^1Z \lrcorner \mathcal{L}) + \int_{\mathcal{U}} (j^1\phi)^* j^1Z \lrcorner d\mathcal{L}
\end{aligned} \tag{B.2.1}$$

The last line has been obtained thanks to the Cartan formula for Lie derivative $L_X f = d(X\lrcorner f) + X\lrcorner df$ and the Stokes theorem $\int_{\mathcal{U}} df = \int_{\partial\mathcal{U}} f$.

In order to obtain the first integral in the expression (B.2.1), we need to introduce some useful notation

$$\begin{aligned}
&\mathrm{d}^n x_\mu = \vec{\partial}_\mu \lrcorner \omega = (-1)^{\mu+1}\mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^{\mu-1} \wedge \mathrm{d}x^{\mu+1} \wedge ... \wedge \mathrm{d}x^{n+1} \\
&\mathrm{d}x^\nu \wedge \mathrm{d}^n x_\mu = \omega\delta_\mu^\nu
\end{aligned}$$

Moreover, notice that $\mathrm{d}x^\mu \wedge \omega = 0$, hence $\mathrm{d}y^A \wedge \omega = \theta^A \wedge \omega$, and let us define $\zeta^A = \theta^A(j^1Z) = \beta^A - v_\mu^A \alpha^\mu$. We compute $j^1Z \lrcorner \mathrm{d}\mathcal{L}$:

$$j^1Z \lrcorner \mathrm{d}\mathcal{L} = j^1Z \lrcorner (d\pounds \wedge \omega)$$

$$= j^1 Z \lrcorner \left(\frac{\partial \pounds}{\partial y^A} \theta^A \wedge \omega + \frac{\partial \pounds}{\partial v^A_\mu} \mathrm{d}v^A_\mu \wedge \omega \right)$$
$$= \frac{\partial \pounds}{\partial y^A} \left(\theta^A (j^1 Z) \omega - \theta^A \wedge \omega (j^1 Z) \right) + \frac{\partial \pounds}{\partial v^A_\mu} \left(\mathrm{d}v^A_\mu (j^1 Z) \omega - \mathrm{d}v^A_\mu \wedge \omega (j^1 Z) \right)$$
$$= \frac{\partial \pounds}{\partial y^A} \left(\zeta^A \omega - \theta^A \wedge \alpha^\mu \mathrm{d}^n x_\mu \right) + \frac{\partial \pounds}{\partial v^A_\mu} \left(\left(\frac{\partial \zeta^A}{\partial x^\mu} + v^B_\mu \frac{\partial \zeta^A}{\partial y^B} \right) \omega - \mathrm{d}v^A_\mu \wedge \alpha^\lambda \mathrm{d}^n x_\lambda \right)$$

We have

$$\frac{\partial \zeta^A}{\partial x^\mu} \omega = \frac{\partial \zeta^A}{\partial x^\nu} \delta^\nu_\mu \omega = \frac{\partial \zeta^A}{\partial x^\nu} \mathrm{d}x^\nu \wedge \mathrm{d}^n x_\mu = \left(\mathrm{d}\zeta^A - \frac{\partial \zeta^A}{\partial y^B} \mathrm{d}y^B - \frac{\partial \zeta^A}{\partial v^B_\lambda} \mathrm{d}v^B_\lambda \right) \wedge \mathrm{d}^n x_\mu$$
$$= \left(\mathrm{d}\zeta^A - \frac{\partial \zeta^A}{\partial y^B} \mathrm{d}y^B + \alpha^\lambda \mathrm{d}v^B_\lambda \right) \wedge \mathrm{d}^n x_\mu$$

and

$$\frac{\partial \pounds}{\partial v^A_\mu} \mathrm{d}\zeta^A \wedge \mathrm{d}^n x_\mu = \mathrm{d} \left(\zeta^A \frac{\partial \pounds}{\partial v^A_\mu} \mathrm{d}^n x_\mu \right) - \zeta^A \mathrm{d} \left(\frac{\partial \pounds}{\partial v^A_\mu} \right) \wedge \mathrm{d}^n x_\mu$$

Using the equality $\omega = \frac{1}{n+1} \mathrm{d}x^\mu \wedge \mathrm{d}^n x_\mu$, this yields

$$j^1 Z \lrcorner \mathrm{d}\mathcal{L} = \frac{\partial \pounds}{\partial y^A} \left(\zeta^A \frac{1}{n+1} \mathrm{d}x^\mu - \alpha^\mu \theta^A \right) \wedge \mathrm{d}^n x_\mu + \frac{\partial \pounds}{\partial v^A_\mu} \left(\alpha^\lambda \mathrm{d}v^B_\lambda - \frac{\partial \zeta^A}{\partial y^B} \mathrm{d}y^B \right) \wedge \mathrm{d}^n x_\mu$$
$$+ \mathrm{d} \left(\zeta^A \frac{\partial \pounds}{\partial v^A_\mu} \mathrm{d}^n x_\mu \right) - \zeta^A \mathrm{d} \left(\frac{\partial \pounds}{\partial v^A_\mu} \right) \wedge \mathrm{d}^n x_\mu + \frac{\partial \pounds}{\partial v^A_\mu} \left(v^B_\mu \frac{\partial \zeta^A}{\partial y^B} \mathrm{d}x^\lambda - \alpha^\lambda \mathrm{d}v^A_\mu \right) \wedge \mathrm{d}^n x_\lambda$$
$$= \mathrm{d} \left(\zeta^A \frac{\partial \pounds}{\partial v^A_\mu} \mathrm{d}^n x_\mu \right) + \zeta^A \left(\frac{1}{n+1} \frac{\partial \pounds}{\partial y^A} \mathrm{d}x^\mu - \mathrm{d} \left(\frac{\partial \pounds}{\partial v^A_\mu} \right) \right) \wedge \mathrm{d}^n x_\mu$$
$$- \left(\frac{\partial \pounds}{\partial v^A_\mu} \frac{\partial \zeta^A}{\partial y^B} + \alpha^\mu \frac{\partial \pounds}{\partial y^B} \right) \theta^B \wedge \mathrm{d}^n x_\mu + \alpha^\lambda \frac{\partial \pounds}{\partial v^A_\mu} \left(\mathrm{d}v^B_\lambda \wedge \mathrm{d}^n x_\mu - \mathrm{d}v^A_\mu \wedge \mathrm{d}^n x_\lambda \right)$$

By construction of $j^1\phi$, $(j^1\phi)^* \theta^B = 0$, implying

$$(j^1\phi)^* \left(\left(\frac{\partial \pounds}{\partial v^A_\mu} \frac{\partial \zeta^A}{\partial y^B} + \alpha^\mu \frac{\partial \pounds}{\partial y^B} \right) \theta^B \wedge \mathrm{d}^n x_\mu \right) = 0$$

Furthermore

$$(j^1\phi)^* \left(\mathrm{d}v^B_\lambda \wedge \mathrm{d}^n x_\mu \right) = \left((j^1\phi)^* \mathrm{d}v^B_\lambda \right) \wedge \left((j^1\phi)^* \mathrm{d}^n x_\mu \right) = \mathrm{d} \left((j^1\phi)^* v^B_\lambda \right) \wedge \mathrm{d}^n x_\mu$$
$$= \mathrm{d} \left(\frac{\partial \phi^A}{\partial x^\lambda} \right) \wedge \mathrm{d}^n x_\mu = \frac{\partial}{\partial x^\mu} \left(\frac{\partial \phi^A}{\partial x^\lambda} \right) \mathrm{d}x^\mu \wedge \mathrm{d}^n x_\mu$$
$$= \frac{\partial^2 \phi^A}{\partial x^\mu \partial x^\lambda} \omega$$

Hence

$$(j^1\phi)^* \left(\mathrm{d}v^B_\lambda \wedge \mathrm{d}^n x_\mu - \mathrm{d}v^B_\mu \wedge \mathrm{d}^n x_\lambda \right) = \left(\frac{\partial^2 \phi^A}{\partial x^\mu \partial x^\lambda} - \frac{\partial^2 \phi^A}{\partial x^\lambda \partial x^\mu} \right) \omega = 0$$

since $\phi \in \mathscr{C}^\infty(\mathcal{U}, E)$.

Using those equalities, and applying once again the Stokes theorem, the variation of action can now be written

$$\delta\mathcal{A} = \int_{\partial\mathcal{U}} (j^1\phi)^* \left(j^1 Z \lrcorner \mathcal{L} + \zeta^A \frac{\partial \pounds}{\partial v^A_\mu} \mathrm{d}^n x_\mu \right) - \int_{\mathcal{U}} (j^1\phi)^* \left(\zeta^A \left(\mathrm{d}\left(\frac{\partial \pounds}{\partial v^A_\mu} \right) - \frac{1}{n+1} \frac{\partial \pounds}{\partial y^A} \mathrm{d}x^\mu \right) \wedge \mathrm{d}^n x_\mu \right)$$

Notice that $\zeta^A = j^1 Z \lrcorner \theta^A$ and

$$(j^1\phi)^*((j^1 Z \lrcorner \theta^A) \wedge \mathrm{d}^n x_\mu) = (j^1\phi)^*(j^1 Z \lrcorner (\theta^A \wedge \mathrm{d}x_\mu))$$

since $j^1\phi$ is holonomic, this yields that the integral along $\partial\mathcal{U}$ in the expression of $\delta\mathcal{A}$ can be written

$$\int_{\partial\mathcal{U}} (j^1\phi)^* \left(j^1 Z \lrcorner \left(\frac{\partial \pounds}{\partial v^A_\mu} \theta^A \wedge \mathrm{d}^n x_\mu + \mathcal{L} \right) \right)$$

The expression of the variation of action is finally

$$\delta\mathcal{A} = \int_{\partial\mathcal{U}} (j^1\phi)^* \left(j^1 Z \lrcorner \left(\frac{\partial \pounds}{\partial v^A_\mu} \theta^A \wedge \mathrm{d}^n x_\mu + \mathcal{L} \right) \right) - \int_{\mathcal{U}} (j^1\phi)^* \left(\zeta^A \left(\mathrm{d}\left(\frac{\partial \pounds}{\partial v^A_\mu} \right) - \frac{1}{n+1} \frac{\partial \pounds}{\partial y^A} \mathrm{d}x^\mu \right) \wedge \mathrm{d}^n x_\mu \right)$$

B.3 Solutions of the pendulum system

.

We introduce the notation $\omega_0 = \sqrt{\frac{g}{l}}$. Writing E as $2mglk^2$, we can discuss the expression of the solutions with respect to $k \in \mathbb{R}_+$ thanks to the Jacobi elliptic functions for any solution starting at $\theta(0) = 0$. Solution starting at another position can be deduced by inverting the function to obtain the necessary time translation.

$k < 1$ In this case, the pendulum oscillates around the position $\theta = 0$.

$$\begin{aligned} \theta(t) &= sgn\left(\mathrm{sn}(\omega_0 t, k^2)\,\mathrm{dn}(\omega_0 t, k^2)\right) \arccos\left(1 - 2k^2\,\mathrm{sn}^2(\omega_0 t, k^2)\right) \\ \dot{\theta}(t) &= 2k\omega_0\,\mathrm{cn}(\omega_0 t, k^2) \end{aligned}$$

$k = 1$ This is the unstable limit case. The pendulum takes an infinite amount of time to reach the position $\theta = \pi$.

$$\begin{aligned} \theta(t) &= 4\arctan\left(e^{\omega_0 t}\right) - \pi \\ \dot{\theta}(t) &= \frac{2\omega_0}{\cosh(\omega_0 t)} \end{aligned}$$

$k > 1$ In this case the pendulum spins around the origin in circles.

$$\begin{aligned} \theta(t) &= sgn\left(\mathrm{sn}\left(k\omega_0 t, \frac{1}{k^2}\right) \mathrm{cn}\left(k\omega_0 t, \frac{1}{k^2}\right)\right) \arccos\left(1 + 2k^2\left(\mathrm{dn}^2\left(k\omega_0 t, \frac{1}{k^2}\right) - 1\right)\right) \\ \dot{\theta}(t) &= 2k\omega_0\,\mathrm{dn}\left(k\omega_0 t, \frac{1}{k^2}\right) \end{aligned}$$

B.4 Computation of (12.2.1)

We here introduce a result that will be useful for the computing of the variation of action.

Proposition 7. *The Maurer-Cartan 1-form is solution of the Maurer-Cartan equation*

$$\mathrm{d}\lambda + [\lambda, \lambda] = 0$$

Proof. Let X, Y be two arbitrary vector fields on G,

$$\mathrm{d}\lambda(X, Y) = L_X\lambda(Y) - L_Y\lambda(X) - \lambda([X, Y]).$$

In the case of two left invariant vector fields, $L_X\lambda(Y) = L_Y\lambda(X) = 0$ since λ^A is the dual basis of a left invariant basis. Moreover, in that case we also have $\lambda([X, Y]) = [\lambda(X), \lambda(Y)]$, yielding

$$\mathrm{d}\lambda([X, Y]) + [\lambda(X), \lambda(Y)] = 0$$

Since $\mathrm{d}\lambda$ is a 2-form, and the previous equality is true for any pair of left invariant vector fields, it is also true for any pair of vector fields, leading to

$$\mathrm{d}\lambda + [\lambda, \lambda] = 0$$

□

Let us define $\Xi = \vartheta^A(j^1Z) = \beta^A - \xi^A_\mu\alpha^\mu$ so that $\omega(j^1Z) = \alpha^\mu \mathrm{d}^n x_\mu$ and compute $j^1Z\lrcorner\mathrm{d}\ell$:

$$\begin{aligned} j^1Z\lrcorner\mathrm{d}\ell = j^1Z\lrcorner(\mathrm{d}l \wedge \omega) &= j^1Z\lrcorner\left(\frac{\partial l}{\partial y^A}T^A_B\vartheta^B \wedge \omega + \frac{\partial l}{\partial \xi^A_\mu}\mathrm{d}\xi^A_\mu \wedge \omega\right) \\ &= \frac{\partial l}{\partial y^A}T^A_B\left(\vartheta^B(j^1Z)\omega - \vartheta^B \wedge \omega(j^1Z)\right) + \frac{\partial l}{\partial \xi^A_\mu}\left(\mathrm{d}\xi^A_\mu(j^1Z)\omega - \mathrm{d}\xi^A_\mu \wedge \omega(j^1Z)\right) \\ &= \frac{\partial l}{\partial y^A}T^A_B\left(\Xi^B\omega - \alpha^\mu\vartheta^B \wedge \mathrm{d}^n x_\mu\right) + \frac{\partial l}{\partial \xi^A_\mu}\left(\left(\frac{\partial \Xi^A}{\partial x^\mu} + \xi^C_\mu T^B_C\frac{\partial \Xi^A}{\partial y^B} + [\xi_\mu, \beta]^A\right)\omega - \alpha^\nu\mathrm{d}\xi^A_\nu \wedge \mathrm{d}^n x_\nu\right) \end{aligned}$$

We have

$$\begin{aligned} \frac{\partial \Xi^A}{\partial x^\mu}\omega = \frac{\partial \Xi^A}{\partial x^\nu}\delta^\nu_\mu\omega = \frac{\partial \Xi^A}{\partial x^\nu}\mathrm{d}x^\nu \wedge \mathrm{d}^n x_\mu &= \left(\mathrm{d}\Xi^A - \frac{\partial \Xi^A}{\partial y^B}T^B_C\lambda^C - \frac{\partial \Xi^A}{\partial \xi^B_\nu}\mathrm{d}\xi^B_\nu\right) \wedge \mathrm{d}^n x_\mu \\ &= \mathrm{d}\Xi^A \wedge \mathrm{d}^n x_\mu - \frac{\partial \Xi^A}{\partial y^B}T^B_C\lambda^C \wedge \mathrm{d}^n x_\mu + \alpha^\nu\mathrm{d}\xi^A_\nu \wedge \mathrm{d}^n x_\mu \end{aligned}$$

and

$$\frac{\partial l}{\partial \xi^A_\mu}\mathrm{d}\Xi^A \wedge \mathrm{d}x_\mu = \mathrm{d}\left(\Xi^A\frac{\partial l}{\partial \xi^A_\mu}\mathrm{d}^n x_\mu\right) - \Xi^A\mathrm{d}\left(\frac{\partial l}{\partial \xi^A_\mu}\right) \wedge \mathrm{d}^n x_\mu.$$

Moreover,

$$\begin{aligned} \frac{\partial l}{\partial \xi^A_\mu}[\xi_\mu, \beta]^A = \frac{\partial l}{\partial \xi^A_\mu}[\xi_\mu, \Xi + \xi_\nu\alpha^\nu]^A &= \frac{\partial l}{\partial \xi^A_\mu}\alpha^\nu[\xi_\mu, \xi_\nu]^A + \left\langle\frac{\partial l}{\partial \xi_\mu}, [\xi_\mu, \Xi]\right\rangle \\ &= \frac{\partial l}{\partial \xi^A_\mu}\alpha^\nu[\xi_\mu, \xi_\nu]^A + \left\langle\mathrm{ad}^*_{\xi_\mu}\frac{\partial l}{\partial \xi_\mu}, \Xi\right\rangle = \frac{\partial l}{\partial \xi^A_\mu}\alpha^\nu[\xi_\mu, \xi_\nu]^A + \Xi^A\left(\mathrm{ad}^*_{\xi_\mu}\frac{\partial l}{\partial \xi_\mu}\right)_A \end{aligned}$$

where the co-adjoint operator ad^* is defined by $\langle \pi, [\alpha,\beta]\rangle = \langle \beta, \mathrm{ad}^*_\alpha \pi\rangle$. This yields

$$\begin{aligned}
j^1 Z \lrcorner \mathrm{d}\ell &= \Xi^A \frac{\partial l}{\partial y^B} T^B_A \omega - T^B_C \frac{\partial l}{\partial y^B} \alpha^\mu \vartheta^C \wedge \mathrm{d}^n x_\mu \\
&\quad + \frac{\partial l}{\partial \xi^A_\mu}\left(\mathrm{d}\Xi^A \wedge \mathrm{d}^n x_\mu - \frac{\partial \Xi^A}{\partial y^b} T^B_C \lambda^C \wedge \mathrm{d}^n x_\mu + \alpha^\nu \mathrm{d}\xi^A_\nu \wedge \mathrm{d}^n x_\mu\right) \\
&\quad + T^B_C \frac{\partial l}{\partial \xi^A_\mu}\frac{\partial \Xi^A}{\partial y^B} \xi^C_\mu \mathrm{d}x^\mu \wedge \mathrm{d}^n x_\mu + \frac{\partial l}{\partial \xi^A_\mu}[\xi_\mu, \beta]^A \omega - \frac{\partial l}{\partial \xi^A_\mu}\alpha^\nu \mathrm{d}\xi^A_\mu \wedge \mathrm{d}^n x_\nu \\
&= \mathrm{d}\left(\Xi^A \frac{\partial l}{\partial \xi^A_\mu}\mathrm{d}^n x_\mu\right) - \Xi^A \mathrm{d}\left(\frac{\partial l}{\partial \xi^A_\mu}\right) \wedge \mathrm{d}^n x_\mu + \Xi^A T^B_A \frac{\partial l}{\partial y^B}\omega + \Xi^A \left(\mathrm{ad}^*_{\xi_\mu}\frac{\partial l}{\partial \xi_\mu}\right)_A \omega \\
&\quad + T^B_C\left(\frac{\partial l}{\partial \xi^A_\mu}\frac{\partial \Xi^A}{\partial y^B}(\lambda^C - \vartheta^C) - \frac{\partial l}{\partial \xi^A_\mu}\frac{\partial \Xi^A}{\partial y^B}\lambda^C - \frac{\partial l}{\partial y^B}\alpha^\nu \vartheta^C\right) \wedge \mathrm{d}^n x_\mu \\
&\quad + \frac{\partial l}{\partial \xi^A_\mu}\alpha^\nu \left(\mathrm{d}\xi^A_\nu \wedge \mathrm{d}^n x_\mu - \mathrm{d}\xi^A_\mu \wedge \mathrm{d}^n x_\nu + [\xi_\mu, \xi_\nu]^A \omega\right) \\
&= \mathrm{d}\left(\Xi^A \frac{\partial l}{\partial \xi^A_\mu}\mathrm{d}^n x_\mu\right) - \left(\Xi^A\left(\mathrm{d}\left(\frac{\partial l}{\partial \xi^A_\mu}\right) \wedge \mathrm{d}^n x_\mu - \left(\mathrm{ad}^*_{\xi_\nu}\frac{\partial l}{\partial \xi_\nu}\right)_A \omega - T^B_A \frac{\partial l}{\partial y^B}\omega\right)\right) \\
&\quad - T^B_C\left(\frac{\partial l}{\partial \xi^A_\mu}\frac{\partial \Xi^A}{\partial y^B} + \frac{\partial l}{\partial y^B}\alpha^\mu\right)\vartheta^C \wedge \mathrm{d}^n x_\mu \\
&\quad + \frac{\partial l}{\partial \xi^A_\mu}\alpha^\nu \left(\mathrm{d}\xi^A_\nu \wedge \mathrm{d}^n x_\mu - \mathrm{d}\xi^A_\mu \wedge \mathrm{d}^n x_\nu + [\xi_\mu, \xi_\nu]^A \omega\right)
\end{aligned}$$

By construction, $(j^1\phi)^*\vartheta^C = 0$, implying

$$(j^1\phi)^*\left(T^B_C\left(\frac{\partial l}{\partial \xi^A_\mu}\frac{\partial \Xi^A}{\partial y^B} + \frac{\partial l}{\partial y^B}\alpha^\mu\right)\vartheta^C \wedge \mathrm{d}^n x_\mu\right) = 0.$$

We also have

$$\begin{aligned}
\phi^*\mathrm{d}\lambda(\vec{\partial}_\mu, \vec{\partial}_\nu) &= (j^1\phi)^*\mathrm{d}(\xi_\eta \mathrm{d}x^\eta)(\vec{\partial}_\mu, \vec{\partial}_\nu) = \mathrm{d}((j^1\phi)^*\xi_\eta) \wedge \mathrm{d}x^\eta(\vec{\partial}_\mu, \vec{\partial}_\mu) \\
&= \frac{\partial((j^1\phi)^*\xi_\eta)}{\partial x_\rho}\mathrm{d}x^\rho \wedge \mathrm{d}x^\eta(\vec{\partial}_\mu, \vec{\partial}_\nu) = \frac{\partial((j^1\phi)^*\xi_\mu)}{\partial x_\mu} - \frac{\partial((j^1\phi)^*\xi_\nu}{\partial \xi_\nu}
\end{aligned}$$

and, since by proposition (7) λ is solution of the Maurer-Cartan equation,

$$\begin{aligned}
\phi^*\mathrm{d}\lambda(\vec{\partial}_\mu, \vec{\partial}_\nu) &= -[\phi^*\lambda, \phi^*\lambda](\vec{\partial}_\mu, \vec{\partial}_\nu) \\
&= -[(j^1\phi)^*\xi_\mu, (j^1\phi)^*\xi_\nu]
\end{aligned}$$

Therefore, the last term in the expression of $j^1 Z \lrcorner \mathrm{d}\ell$ is cancelled when evaluated along $(j^1\phi)$, that is

$$\begin{aligned}
&(j^1\phi)^*\left(\mathrm{d}\xi^A_\nu \wedge \mathrm{d}^n x_\mu - \mathrm{d}\xi^A_\mu \wedge \mathrm{d}^n x_\nu + [\xi_\mu, \xi_\nu]^A \omega\right) \\
&\quad = \frac{\partial((j^1\phi)^*\xi_\nu)^A}{\partial x_\mu}\mathrm{d}x^\mu \wedge \mathrm{d}^n x_\mu - \frac{\partial((j^1\phi)^*\xi_\mu)^A}{\partial x_\nu}\mathrm{d}x^\nu \wedge \mathrm{d}^n x_\nu + [(j^1\phi)^*\xi_\mu, (j^1\phi)^*\xi_\nu]^A \omega \\
&\quad = \left(\left(\frac{\partial((j^1\phi)^*\xi_\nu)}{\partial x_\mu} - \frac{\partial((j^1\phi)^*\xi_\mu)}{\partial x_\nu}\right) - [(j^1\phi)^*\xi_\mu, (j^1\phi)^*\xi_\nu]\right)^A \omega \\
&\quad = 0
\end{aligned}$$

Hence,

$$(j^1\phi)^* j^1 Z \lrcorner \mathrm{d}\ell = (j^1\phi)^*\left(\mathrm{d}\left(\Xi^A \frac{\partial l}{\partial \xi^A_\mu}\mathrm{d}^n x_\mu\right) - \left(\Xi^A\left(\mathrm{d}\left(\frac{\partial l}{\partial \xi^A_\mu}\right) \wedge \mathrm{d}^n x_\mu - \left(\mathrm{ad}^*_{\xi_\nu}\frac{\partial l}{\partial \xi_\nu}\right)_A \omega - T^B_A \frac{\partial l}{\partial y^B}\omega\right)\right)\right)$$

Applying the Stoke's theorem again in the second integral of the variation of action, $\delta\mathcal{A}$ can now be written

$$\delta\mathcal{A} = \int_{\partial\mathcal{U}} (j^1\phi)^* \left(j^1 Z \lrcorner \ell + \Xi^A \frac{\partial l}{\partial \xi^A_\mu} \mathrm{d}^n x_\mu \right) - \int_{\mathcal{U}} (j^1\phi)^* \left(\Xi^A \left(\mathrm{d} \left(\frac{\partial l}{\partial \xi^A_\mu} \right) \wedge \mathrm{d}^n x_\mu - \left(\mathrm{ad}^*_{\xi_\nu} \frac{\partial l}{\partial \xi_\nu} \right)_A \omega - T^B_A \frac{\partial l}{\partial y^B} \omega \right) \right)$$

C Implementation of the Reissner beam integrator

C++ implementation A first implementation of the algorithms of part VII has been done in C++, in order to take advantage of the object-oriented paradigm.

The `main.cpp` file implements the time and space resolution with the same level of abstraction as the one of algorithm1. To do so, implementation of classes `Group`, `Algebra` and `Operator` must be provided. In the case of the Reissner beam, the template classes `Lie::SE3::Group<T>`, `Lie::SE3::Algebra<T>` and `Lie::SE3::Operator<T>` have been implemented, where `T` is the template type of variables handled by the group; here we use $\mathbb{R}$, typically represented by `float` or `double` types. The solving of χ or ϵ (depending on the type of integrator) has been done by using GSL (GNU Scientific Library) which provides efficient general purpose solvers implemented in C.

The computation of the result is currently not fast enough to be done in real-time for a "good enough" time step (for example $44100Hz$), however no particular optimisations has been done yet, that could significantly improve the computation speed. Without going to much into details, one could mention several ideas to improve the implementation :

– The group elements have been implemented using 3×3 rotation matrices, but taking advantage of the quaternion representation could diminish the number of stored coefficients from 9 to 4, and get rid of the matrix products by replacing it by the faster quaternion multiplication.

– Some specific classes and representation of the Eigen C++ library, used to implement matrices and their operations, could also improve the speed of computation, in particular in terms of matrix products. The error order of the solving of χ or ϵ could also be a parameter on which one could find a compromise between precision and speed.

– The re-usability of the main algorithm, achieved by using template classes, may be to some extent heavier than a direct implementation of the main algorithm for each specific Lie group, and thus slower.

Even if no use has been made of the re-usability of the main algorithm in the case of other group, algebra and operators, one can imagine the interest of such a design to offer an easy implementation of the same algorithm for other 2-dimensional base space problems. Instead of having to write the entire algorithm for each problem, the user would only need to provide valid implementations of the abstract classes related to the Lie group he intends to use the algorithm on, and change the following type aliases :

```
using Group    = Lie::SE3::Group<double>;
using Algebra  = Lie::SE3::Algebra<double>;
using Operator = Lie::SE3::Operator<double>;
```

This implementation being a prototype used to check the validity of the numerical methods on a specific case, here is of course not the place to discuss extensively of the problematic of the implementation of a library; nonetheless, it gives a hint of a possible design of implementation of the methods that could be pursued in the context of a more ambitious work.

Here is the help display of the `reissner` program implementing the algorithm :

```
NAME
    reissner - Computes the position of a Reissner beam on the given 2D space-
               time set

SYNOPSIS
    reissner [OPTION]
```

```
DESCRIPTION
    -o, --output [FILE]
        Output file in which to write. If none given, results are output on
        standard output
    -i, --time-integrator
        Use a time integration algorithm to compute the result (default)
    -I, --space-integrator
        Use a space integration algorithm to compute the result
    -T, --time-number [INTEGER]
        Uses the given value as the number of time steps, default 6001
    -S, --space-number [INTEGER]
        Uses the given value as the number of space steps, default 11
    -t, --time-step [FLOAT]
        Uses the given value as a time step, default 0.0005
    -s, --space-step [FLOAT]
        Uses the given value as a space step, default 0.1
    --time-resample [INTEGER]
        Resamples the output in time with the given number of samples
    --space-resample [INTEGER]
        Resamples the output in space with the given number of samples
    --young [FLOAT]
        Young modulus value, default 5000
    --poisson [FLOAT]
        Poisson ratio value, default 0.35
    --density [FLOAT]
        Density of the material, default 1000
    --radius  [FLOAT]
        Radius of the circular cross-section of the beam, default 0.1
    -v, --verbose
        Verbose output
    -h, --help
        Prints this message and exits
```

MATLAB implementation Since the solver used in the `C++` implementation did not converge (see remark 18.2), the algorithm has been reimplemented with MATLAB to use the larger and more high-level library of solvers available. However, the resulting algorithm does not perform significantly better to solve the non linear system, and runs much slower than the `C++` implementation.

References

[1] Joël Bensoam and Florie-Anne Baugé. Multisymplectic geometry and covariant formalism for mechanical systems with a Lie group as configuration space: application to the Reissner beam. 2017.

[2] Joël Bensoam, Frédéric Hélein, and Pierre Carré. Differential Geometry applied to Acoustics. Non Linear Propagation in Reissner Beams : an integrable system? 2017.

[3] F. Demoures, F. Gay-Balmaz, M. Kobilarov, and T. Ratiu. Multisymplectic Lie group variational integrators for a geometrically exact beam. *ArXiv Mathematics e-prints*, 2016.

[4] F. Demoures, F. Gay-Balmaz, and T. Ratiu. Multisymplectic variational integrators and space/time symplecticity. *ArXiv Mathematics e-prints*, 2013.

[5] M. Desbrun, A. N. Hirani, M. Leok, and J. E. Marsden. Discrete Exterior Calculus. *ArXiv Mathematics e-prints*, 2005.

[6] Mathieu Desbrun, Eva Kanso, and Yiying Tong. Discrete Differential Forms for Computational Modelling, 2009.

[7] Ernst Hairer, Christian Lubich, and Gerhard Wanner. *Geometric Numerical Integration : Structure Preserving Algorithmsfor Ordinary Differential Equations*. Springer, 2006.

[8] D D Holm. Geometric mechanics, part ii: Rotating, translating and rolling, 2011.

[9] Melvin Leok and Tatiana Shingel. General techniques for constructing variational integrators, 2011.

[10] M. Mañas. *The Principal chiral model as an integrable system*, pages 147–173. Vieweg Teubner Verlag, 1994.

[11] J. E. Marsden and M. West. Discrete Mechanics and Variational Integrators, 2001.

[12] Jerrold E. Marsden, George W. Patrick, and Steve Shkoller. Multisymplectic Geometry, Variational Integrators, and Nonlinear PDEs. *Comm. Math. Phys*, 199:351–395.

[13] A. Echeverría-Enríquez M. C. Muñoz-Lec and N. Román-Roy. Geometry of multisymplectic Hamiltonian first-order field theories. *Journal of Mathematics and Physics*, page 7402–7444, 2000.

[14] Ari Stern, Yiyig Tong, Mathieu Desbrun, and Jerrold E. Marsden. Geometric Computational Electrodynamics with Variational Integrators and Discrete Differential Forms. 2009.

[15] J. Vankerschaver. Continuous and discrete aspects of Lagrangian field theory with nonholonomic constraints, 2006.